Introduction to Algorithms and Data Structures

9. Sorting (2): Merge sort, quick sort, analysis, and counting sort

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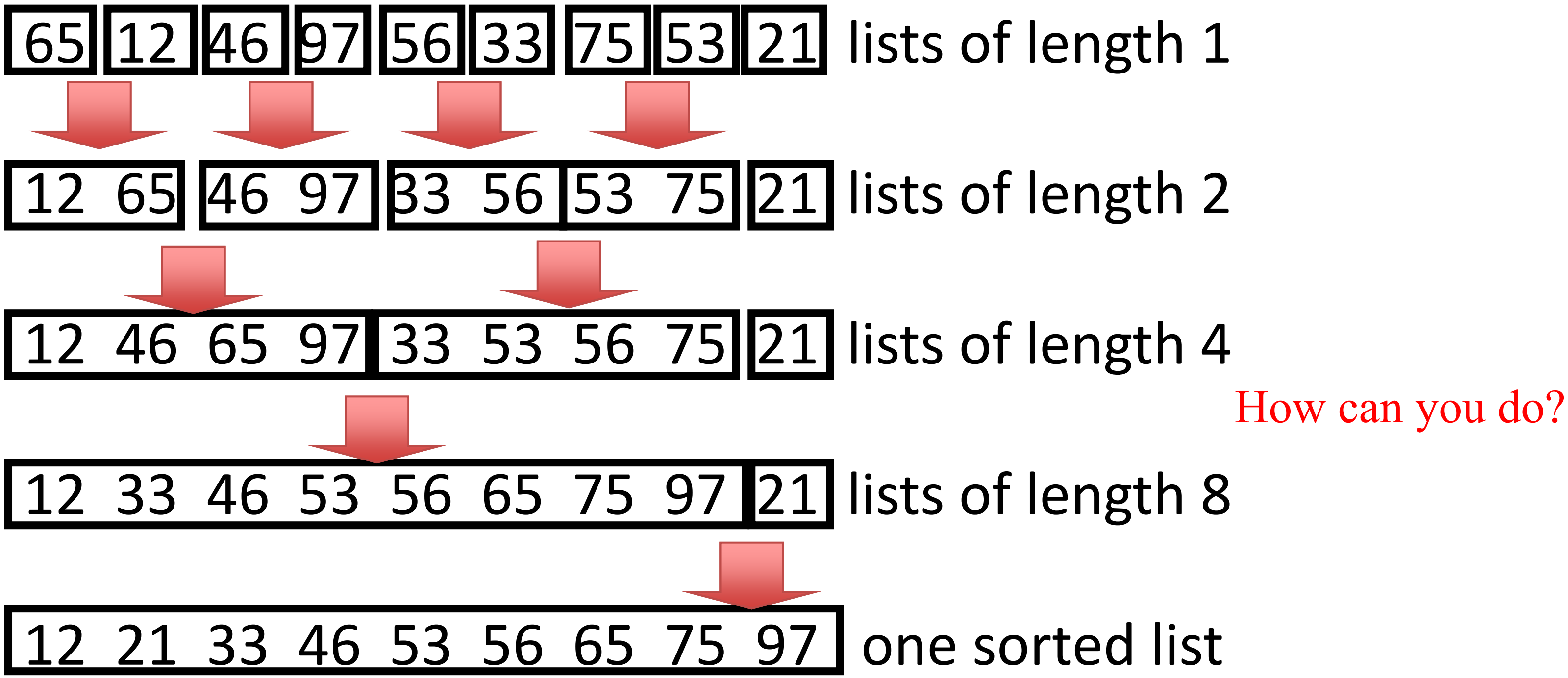
John von Neumann
1903−1957

MERGE SORT

Merge sort

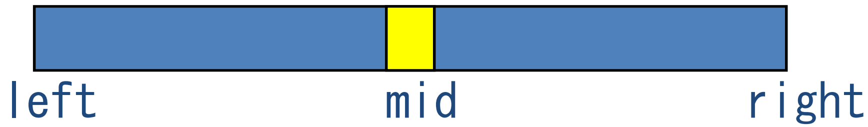
- It repeats to merge two sorted lists into one (sorted) list



- First, it repeats to divide until all lists have length 1, and next, it merges each two of them.

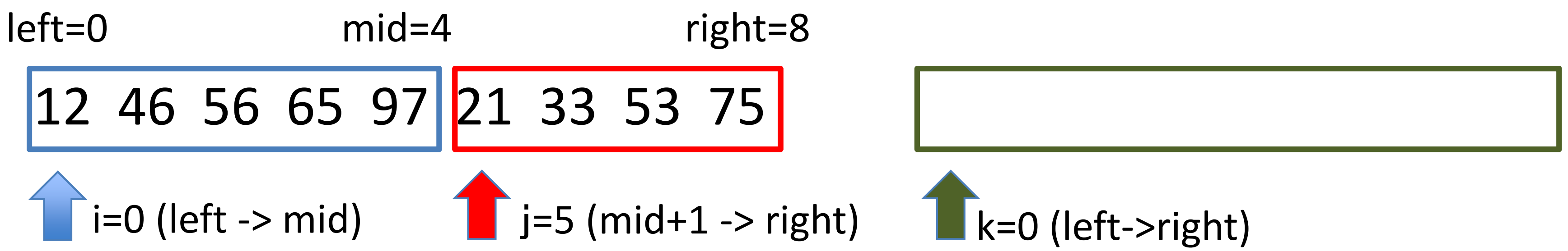
Implementation of merge sort: Typical recursive calls

- The interval that will be sorted: [left, right]
- Find center mid = (left + right)/2

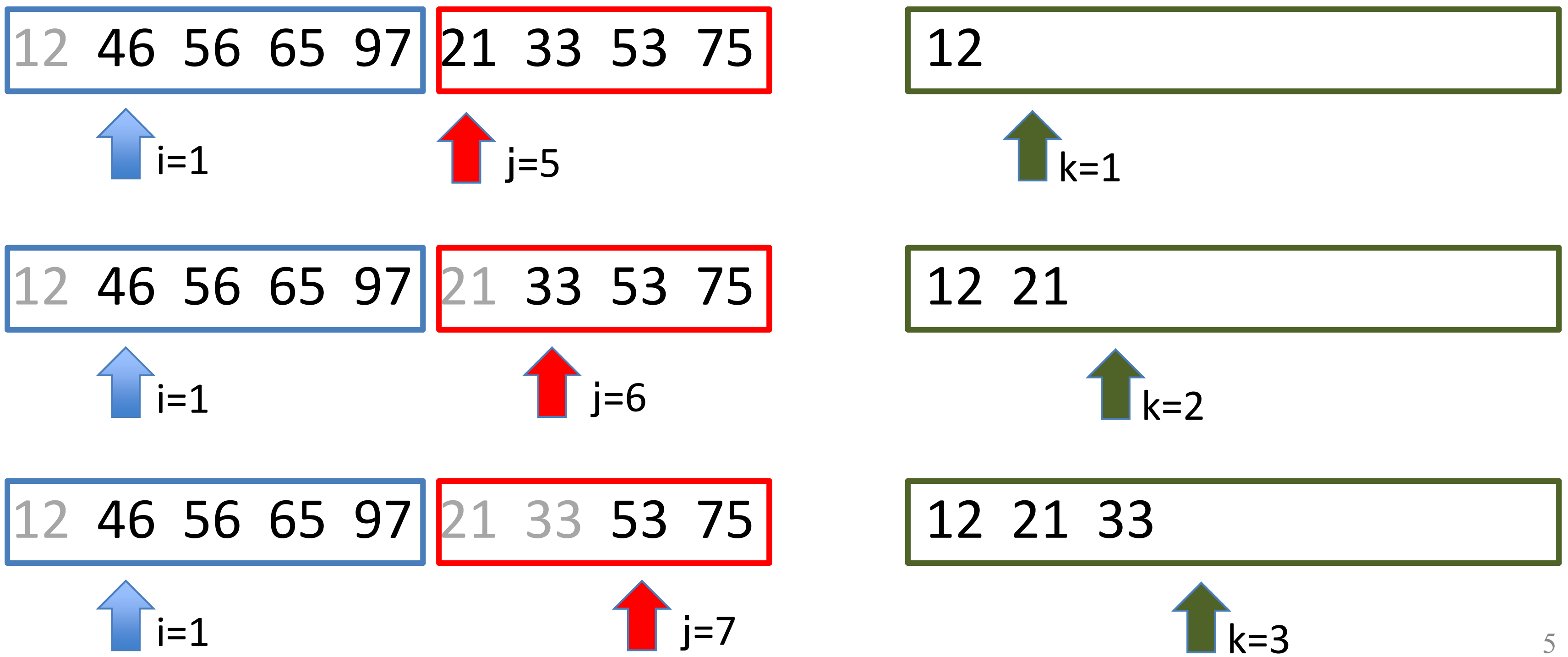


- [left,right]➔[left,mid], [mid+1,right]
- Perform merge sort for each of them, and merge these sorted lists into one sorted list.

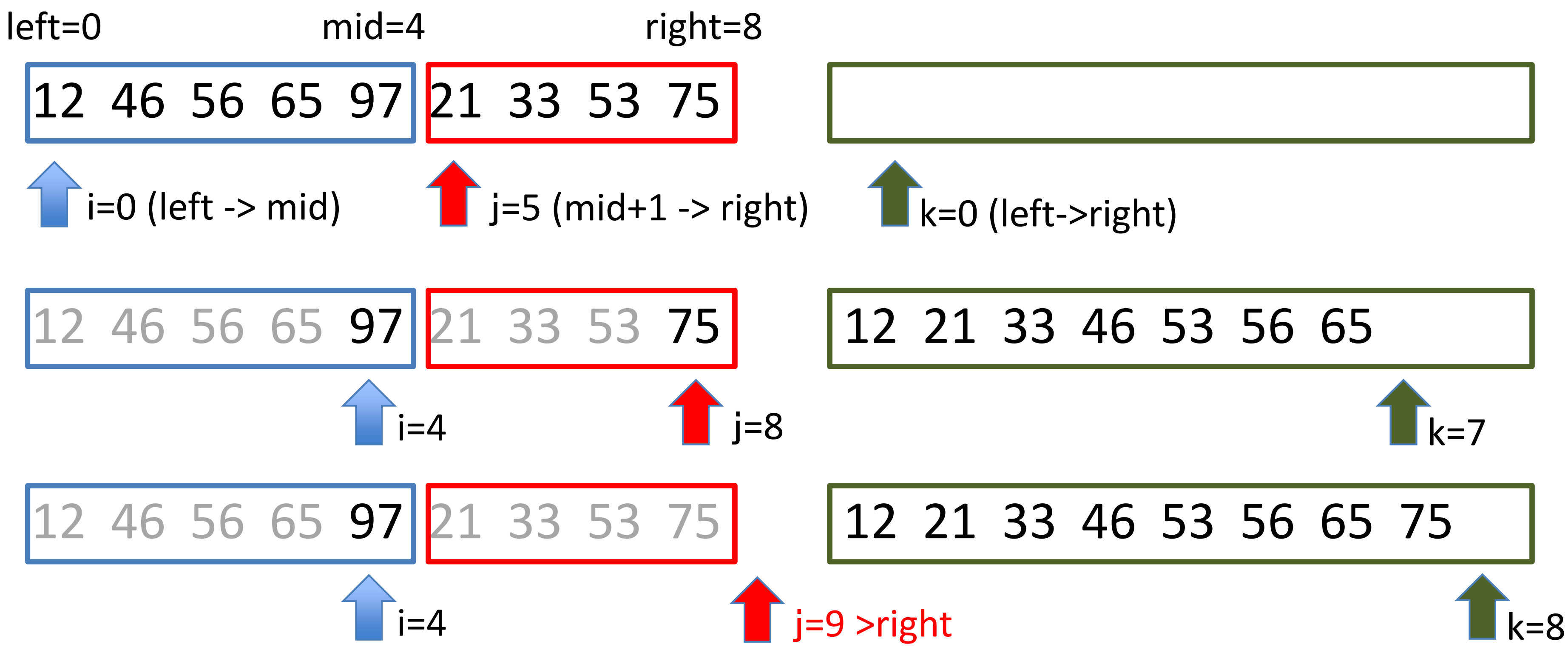
How to merge?



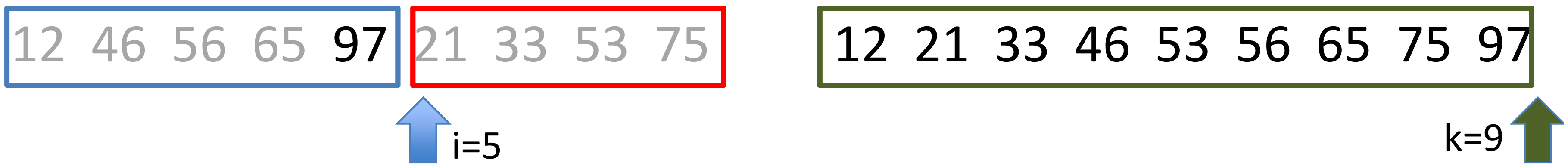
Between 2 tops of 2 sequences, move smaller one to the new array



How to merge?



When one sequence is empty (i>mid or j>right), copy the others



Task takes right-left+1 steps

Outline of merge sort

```
MergeSort(int left, int right){
    int mid;
    if(interval [left,right] is short)
      (sort by any other simple sort algorithm);
    else{
      mid = (left+right)/2;
      MergeSort(left, mid);
      MergeSort(mid+1, right);
      Merge [left, mid] and [mid+1, right];
    }
}
```

We can merge two lists of length p and q in $O(p+q)$ time.

Implementation of merging

We need to merge [left, mid] and [mid+1, right] efficiently

Top of left   top of right   index of new array

```
i=left; j=mid+1; k=left;
while(i<=mid && j<=right)
  if(a[i] <= a[j]) {
    b[k]=a[i]; k++; i++:
  } else {
    b[k]=a[j]; k++; j++;
  }
while(j<=right){ b[k]=a[j]; k++; j++; }
while(i<=mid){ b[k]=a[i]; k++; i++; }
for(i=left; i<=right; i++) a[i]=b[i];
```

$O(p+q)$ (the first while loop)

Put the smaller one of two tops into b[]

Copy remainders of the non-empty list to b[]

Write back to a[] from b[]

Merge sort: Time complexity

- $T(n)$: Time for merge sort on n data
  - $T(n) = 2T(n/2) +$ "time to merge"
    $= 2T(n/2) + cn + d$ (c, d: some positive constant)
- To simplify, letting $n = 2^k$ for integer k,

$$
\begin{aligned}
T(2^k) &= 2T(2^{k-1}) + c2^k + d \\
&= 2(2T(2^{k-2}) + c2^{k-1} + d) + c2^k + d \\
&= 2^2T(2^{k-2}) + 2c2^k + (1+2)d \\
&= 2^2(2T(2^{k-3}) + c2^{k-2} + d) + 2c2^k + (1+2)d \\
&= 2^3T(2^{k-3}) + 3c2^k + (1+2+4)d \\
&\vdots \\
&= 2^iT(2^{k-i}) + ic2^k + (1+2+\ldots 2^{i-1})d \\
&= 2^kT(2^0) + kc2^k + (1+2+\ldots 2^{k-1})d \\
&= bn + cn\log n + (n-1)d \in O(n\log n)
\end{aligned}
$$

Merge sort: Space complexity

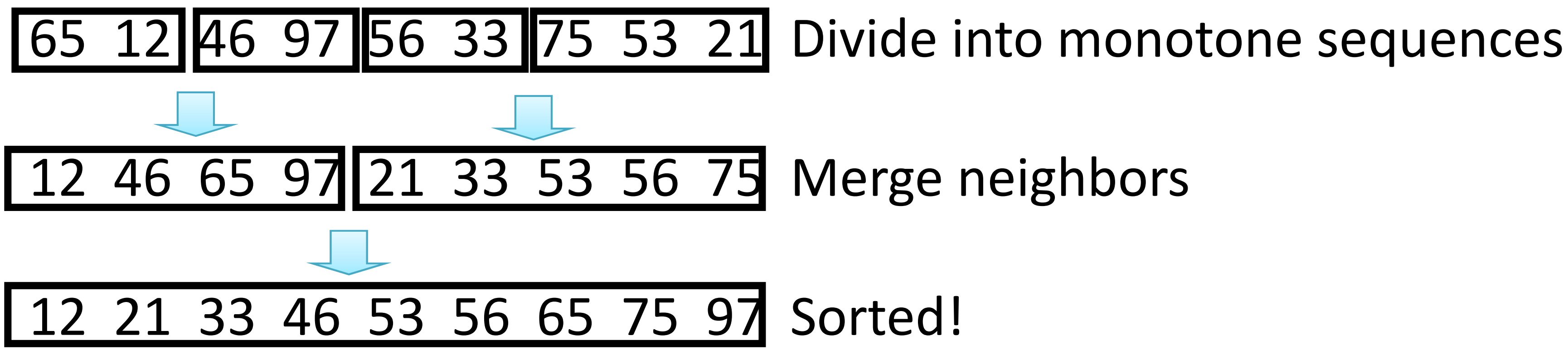
- It is easy to implement by using two arrays a[] and b[].
  - Thus space complexity is Θ(n), or we need *n* extra array for b[].
  - It seems to be difficult to remove this “extra” space.
  - On the other hand, we can omit “Write back b[] to a[]” (in the 2 previous slides) when we use a[] and b[] alternately.

Maybe this “extra” space is the reason why merge sort is not used so often…

Monotone sequence merge sort

- Bit improved merge sort from the practical viewpoint.
- It first divides input into monotone sequences and merge them. (Original merge sort does not check the input)

Example: For 65, 12, 46, 97, 56, 33, 75, 53, 21;



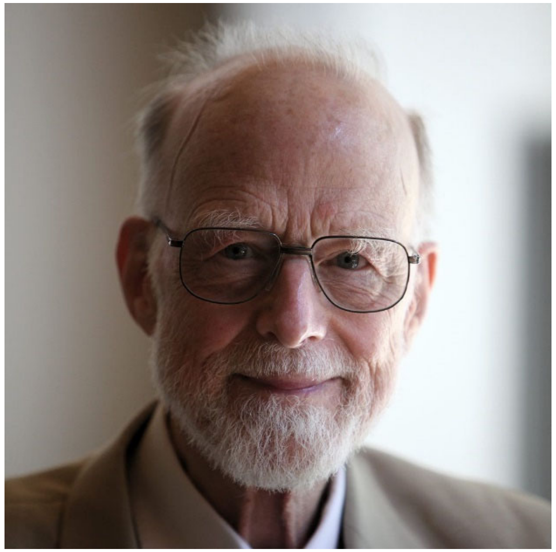
Monotone sequence merge sort: Time complexity

- We can merge in O(*p*+*q*) time to merge two sequences of length *p* and *q*
- After merging, the number of sequences becomes in half.
  - When the number of monotone sequences is h, the number of recursion is $\log_2 h$ times.
- One recursion takes O(*n*) time

  → O(*n* log *h*) time in total.

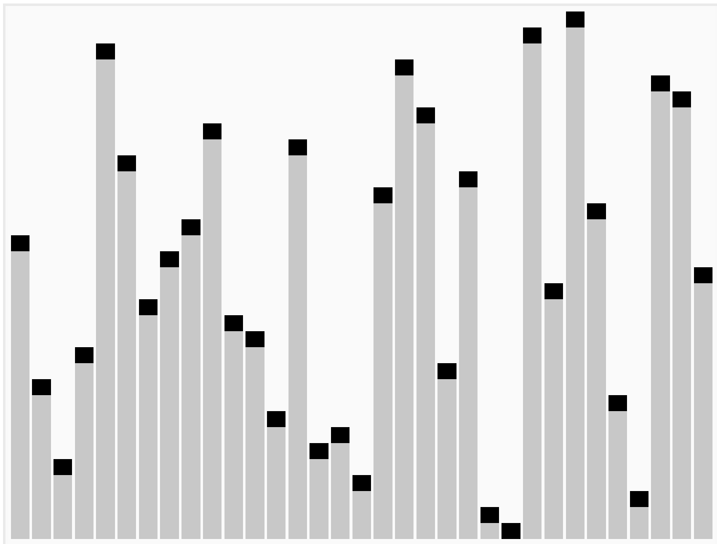
- When data is already sorted: *h* = 1 → O(*n*) time
- The maximum number of monotone sequences is n/2

  → O(*n* log *n*) time in total.



Tony Hoare
1934–

QUICK SORT



C.A.R. Hoare, “Algorithm 64: Quicksort”.
Communications of the ACM 4 (7): 321 (1961)

Quick sort

- Main property: On average, the fastest sort!
- Outline of quick sort:
  - Step 1: Choose an element x (which is called pivot)
  - Step 2: Move all elements ≦ x to left
    Move all elements ≧ x to right

  - Step 3: Sort left and right sequences independently and recursively
    - (When sequence is short enough, sort by any simple sorting)

Quick sort: Example
Step 1. Choose an element x

- Sort the following array by quick sort:

65	12	46	97	56	33	75	53	21

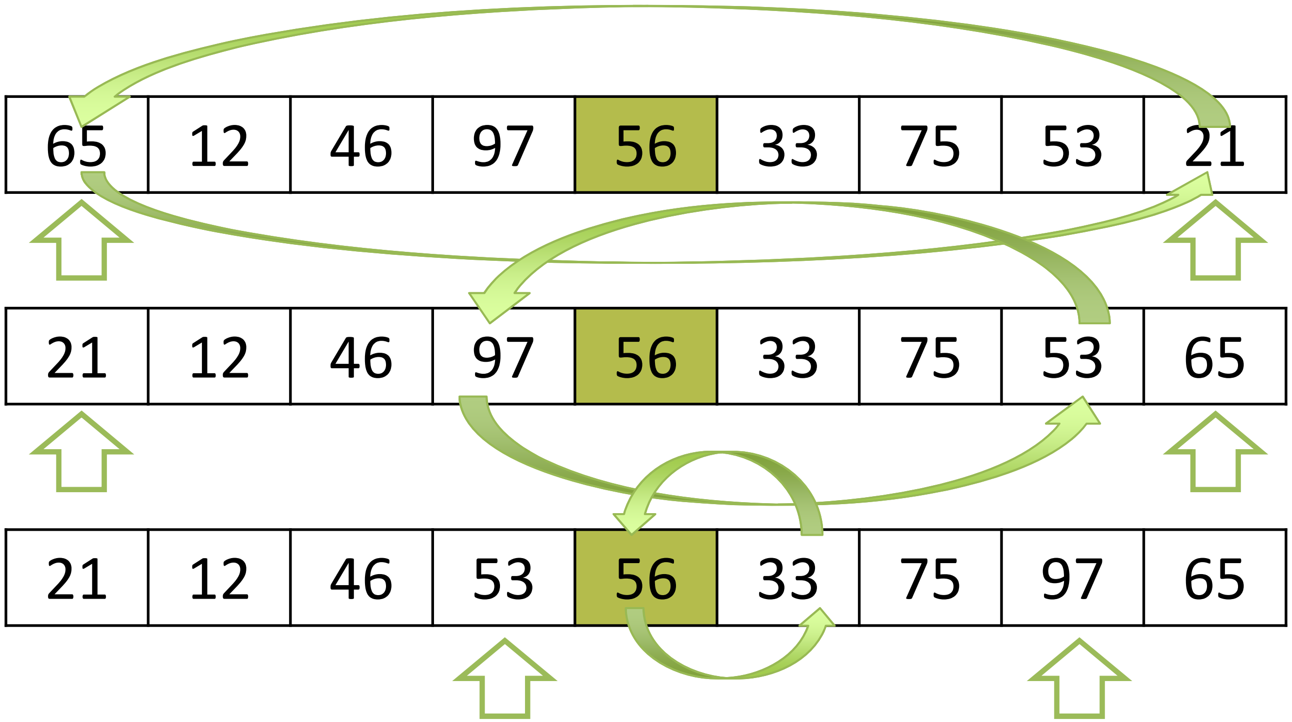
- Choose x=56, for example;

65	12	46	97	56	33	75	53	21

Quick sort: Example
Step 2. Move element w.r.t x:

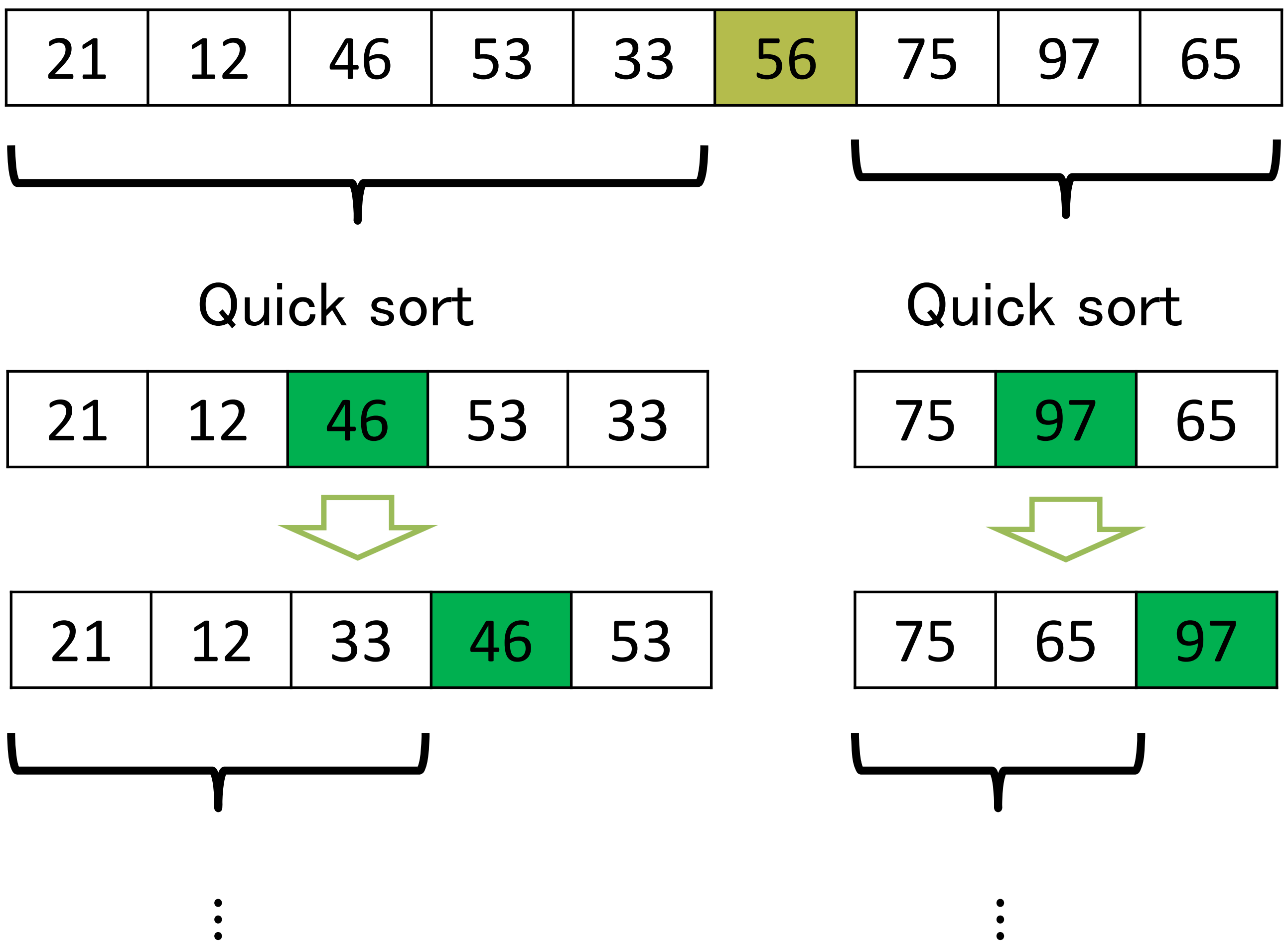
- 

- Start from [l, r] = [0,n-1], move l and r,
  Swap a[l] and a[r] when a[l] >= x && a[r] < x



Quick sort: Example

Step 3. Sort left and right sequences recursively



Quick sort: Program

```
qsort(int a[], int left, int right){
  int i, j, x;
  if(right <= left) return;
  i = left; j = right; x = a[(i+j)/2];
  while(i<=j){
    while(a[i]<x) i=i+1;
      while(a[j]>x) j=j-1;
       if(i<=j){
      swap(&a[i], &a[j]); i=i+1; j=j-1;
    }
  }
  qsort(a, left, j); qsort(a, i, right);
}
```

Note: In MIT textbook, there is another implementation.

Quick sort: Time complexity
Worst case

- When the pivot x is the maximum or minimum element, we divide
  length n → length 1 + length n-1
- This repeats until the longer one becomes 2
- The number of comparisons; $\sum_{k=2}^{n} k \in \Theta(n^2)$

Almost as same as the bubble sort...

Analysis of QuickSort

– Sorting Problem

Input: An array a[n] of n data

Output: The array a[n] such that

a[1]<a[2]<…<a[n]

★To simplify, we assume that there are no pair **i≠j** with a[i]=a[j]

Analysis of QuickSort

– In practical, QuickSort is said to be “the fastest sort”
  - Representative algorithm based on divide-and-conquer
  - If partition is well-done, it runs in $O(n \log n)$ time.
  - If each partition is the worst case, it runs in $O(n^2)$ time.

…Can we analyze theoretically, and guarantee the running time?

Analysis of QuickSort

- Review of QuickSort
  - Call qsort(a,1,n)
  - If qsort(a, i, j) is called,
    - (Randomly) choose a pivot a[m]
    - Divide a[] into “former” and “latter” by a[m]. I.e., sort as

      a[i’]<a[m] for i≦ i’ < m, and

      a[j’]>a[m] for m< j’< j.
    - Return qsort(a, i, i’), a[m], qsort(a, j’, j) as the result

Analysis of QuickSort

– Though they say that QuickSort is the fastest in a practical sense,,,
  - When a[m] becomes always the center of a[i]..a[j], we have

    $$T(n) \leqq 2T(n/2) + (c+1)\, n$$

    and hence $T(n) = \mathrm{O}(n \log n)$.

  - When a[m] becomes always either a[i] or a[j], we have

    $$T(n) \leqq T(1) + T(n\text{-}1) + (c+1)n$$

    and hence $T(n) = \mathrm{O}(n^2)$.

[C.F.]
We can always find the center in $\mathrm{O}(j\text{-}i)$ time.

What about average case?

Analysis of QuickSort

– They say that QuickSort is the fastest in a practical sense,,,
  - Assumption: each item in a[i] ... a[j] is chosen uniformly at random.
    – Thus the kth largest value is chosen as the pivot with probability $1/(j\text{-}i+1)$

[Theorem] An upper bound of the expected value of the running time of QuickSort is $2n\ H(n) \sim 2n \log n$

H_n is the harmonic number and $H_n = O(\log n)$.

It runs fast since few overhead.

Analysis of QuickSort

[Theorem] An upper bound of the expected value of the running time of QuickSort is $2n\,H(n) \sim 2n \log n$

– Notation

» s_k is the *k*th largest item in a[1]…a[n].

» Define indicator variable X_{ij} as follows

$$X_{ij} = \begin{cases} 0 & s_i \text{ and } s_j \text{ are not compared in the algorithm} \\ 1 & s_i \text{ and } s_j \text{ are compared in the algorithm} \end{cases}$$

– Running time of QuickSort

~ the number of comparisons= $\sum_{i=1}^{n} \sum_{j>i} X_{ij}$

Analysis of QuickSort

[Theorem] An upper bound of the expected value of the running time of QuickSort is $2n\,H(n) \sim 2n \log n$

- The expected value of the running time of QuickSort=

$$E[\sum_{i=1}^{n}\sum_{j>i} X_{ij}] = \sum_{i=1}^{n}\sum_{j>i} E[X_{ij}] \qquad \text{(Linearity of expectation value)}$$

- Define as "p_{ij} : probability that s_i and s_j are compared",

$$E[X_{ij}] = p_{ij} \times 1 + (1 - p_{ij}) \times 0 = p_{ij}$$

  Thus consider the value of p_{ij}

- When s_i and s_j are compared??
  1. One of them is chosen as the pivot, and
  2. They are not yet separated by qsort up to there

     ⇔ Any element between s_i and s_j are not yet chosen as a pivot

Analysis of QuickSort

[Theorem] An upper bound of the expected value of the running time of QuickSort is $2n\,H(n) \sim 2n \log n$

- When s_i and s_j are compared?
  1. One of them is chosen as the pivot, and
  2. They are not yet separated by qsort up to there

     ⇔ Any element between s_i and s_j is not yet chosen as a pivot
  - The ordering of pivots in $s_i,\ s_{i+1},\ s_{i+2},\ \ldots,\ s_{j-1}, s_j$ is uniformly at random!
  - Thus s_i or s_j is the first pivot with probability $\dfrac{2}{j-i+1}$

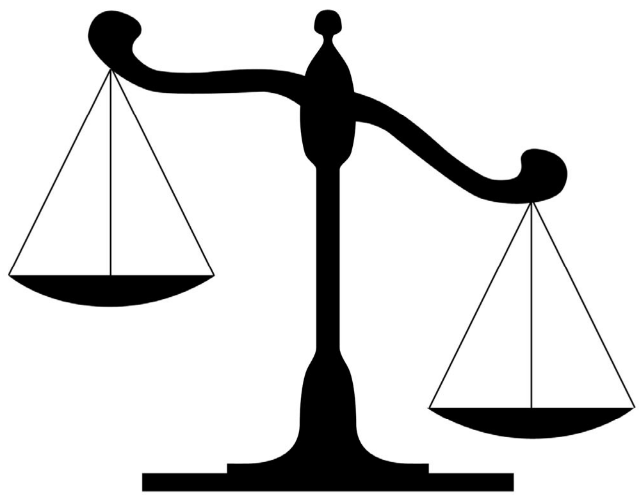
Therefore, the expected time of the running time of QuickSort

$$= E[\sum_{i=1}^{n}\sum_{j>i} X_{ij}] = \sum_{i=1}^{n}\sum_{j>i} E[X_{ij}] = \sum_{i=1}^{n}\sum_{j>i} p_{ij} = \sum_{i=1}^{n}\sum_{j>i} \frac{2}{j-i+1} = \sum_{i=1}^{n}\sum_{k=2}^{n-i+1} \frac{2}{k} \le 2\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{1}{k} = 2nH(n)$$

COMPUTATIONAL COMPLEXITY OF THE SORTING PROBLEM

Sort on Comparison model

- Sort on comparison model: Sorting algorithms that only use the "ordering" of data
  - It only uses the property of "a > b, a = b, or a < b"; in other words, the value of variable is not used.



Computational complexity of sort on comparison model

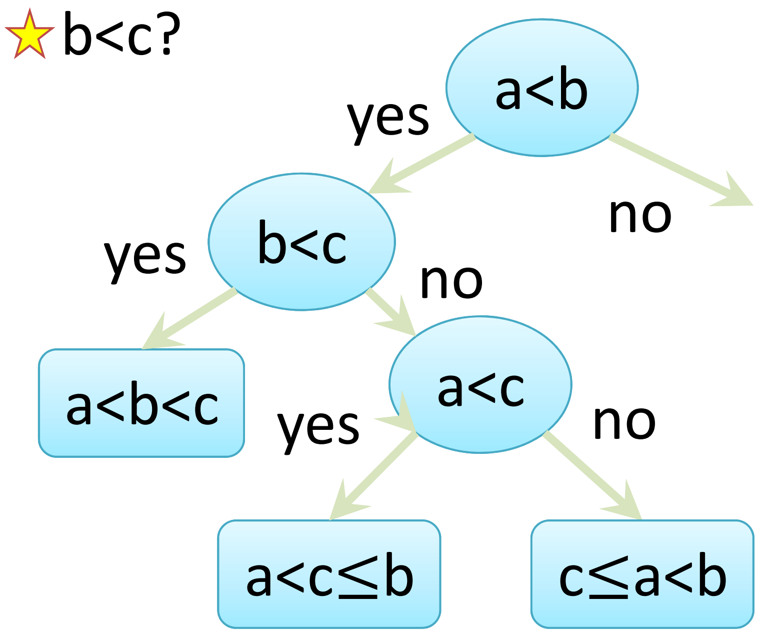
- Upper bound: $O(n \log n)$
  There exist sort algorithms that run in time proportional to $n \log n$ (e.g., merge sort, heap sort, …).
- Lower bound: $\Omega(n \log n)$
  For any comparison sort, there exists an input such that the algorithm runs in time proportional to $n \log n$.

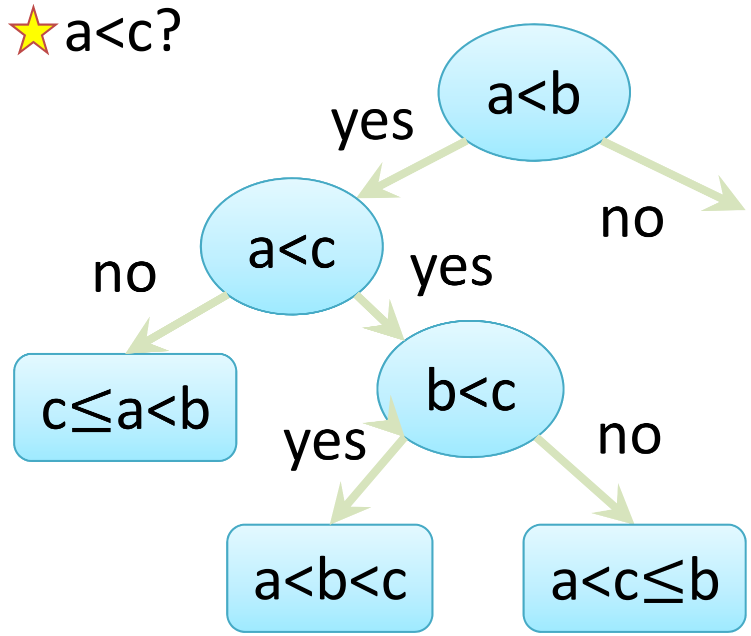
We consider the lower bound of comparison sorting.

Computational complexity of comparison sort: lower bound

- Simple example; sort 3 data a, b, c:

  First, compare (a,b), (b,c), or (c, a). Without loss of generality, we assume that (a,b) is compared; then the next pair is (b,c) or (c,a):





Computational complexity of comparison sort: lower bound

- What we know from sorting of {a, b, c}:
  - For any input, we obtain the solution at most 3 comparison operators.
  - There are some input that we have to compare at least 3 comparison operations.

  = maximum length of a path from root to a leaf is 3,

  which gives us the lower bound.

When we build a decision tree such that "the longest path from root to a leaf is shortest," that length of the longest path gives us a lower bound of sorting problem.

Computational complexity of comparison sort: lower bound

The case when *n* data are sorted

- Let k be the length of the longest path in an optimal decision tree T. Then,

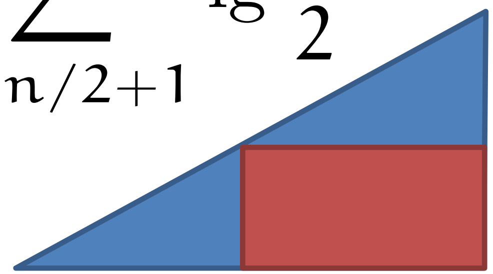
  The number of leaves of T $\leqq 2^k$

- Since all possible permutations of *n* items should appear as leaves, $n! \leqq 2^k$

- By taking logarithm,

$$k = \lg 2^k \geq \lg n! = \sum_{i=1}^{n} \lg i \geq \sum_{i=n/2+1}^{n} \lg \frac{n}{2}$$

$$= \frac{n}{2} \lg \frac{n}{2} \in \Omega(n \log n)$$



Non-comparison sort: Counting sort

- We need some assumption:

  data[i]∈{1,…,k} for 1≦i≦n, k∈O(n)

  (For example, scores of many students)

- Using values of data, it sorts in Θ(*n*) time.

Counting sort

Input: data[i] $\in \{1,\ldots,k\}$ for $1 \leqq i \leqq n$, $k \in O(n)$

Idea: Decide the position of element x

– Count the number of element less than x

➔That number indicates the position of x

Example:

3	7	4	1	2	5

1	2	3	4	5	6	7
1	1	1	1	1	0	1

➔

1	2	3	4	5	6	7
0	1	2	3	4	5	5

⬇

1	2	3	4	5	7

Counting sort

Q. When array contains many data of same values?

A. Use 3 arrays a[], b[], c[] as follows;
(a[]: input, b[]: sorted data, c: counter)

- c[a[i]] counts the number of data equal to a[i]

- For each j with 0≦j≦k,
let c'[j] := c[0] + … + c[j-1] + c[j], then
c'[j] indicates the number of data whose value is less than j

- Copy a[i] to certain b[] according to the value of c'[]

Counting sort: program

```
CountingSort(a, b, k){
  for i=0 to k
     c[i] = 0;

  for j=0 to n-1
     c[ a[j] ] = c[ a[j] ] + 1;

  for i=1 to k
     c[i] = c[i] + c[i-1];

  for j=n-1 downto 0
    b[ c[a[j]]-1 ] = a[j];
    c[a[j]] = c[a[j]] - 1;
}
```

Initialize counter c[]

Count the number of the value in a[i]

Compute c'[] from c[]
In an efficient way!

Copy a[] to b[]

Counting sort: Example
Sort integers (3,6,4,1,3,4,1,4)

- After (2);
  c[]=(0,2,0,2,3,0,1)
- After (3);
  c[]=(0,2,2,4,7,7,8)

a[7]=4 => b[c[4]-1] = b[6], c[4]=6
a[6]=1 => b[c[1]-1] = b[1], c[1]=1
a[5]=4 => b[c[4]-1] = b[5], c[4]=5
a[4]=3 => b[c[3]-1] = b[3], c[3]=3
a[3]=1 => b[c[1]-1] = b[0], c[1]=0
a[2]=4 => b[c[4]-1] = b[4], c[4]=4
a[1]=6 => b[c[6]-1] = b[7], c[6]=7
a[0]=3 => b[c[3]-1] = b[2], c[3]=2

```
CountingSort(a, b, k){
  for i=0 to k
     c[i] = 0;

(2)for j=0 to n-1
     c[ a[j] ] = c[ a[j] ] + 1;

(3)for i=1 to k
     c[i] = c[i] + c[i-1];

  for j=n-1 to downto 0
    b[ c[a[j]]-1 ] = a[j];
    c[a[j]] = c[a[j]] - 1;
}
```

Sort is said to be "stable" when two variables of the same value in order after sorting.

- After ...
  c[]=(0,2, ...,7,7,8)

a[7]=4 => b[c[4]-1] = b[6], c[4]=6
a[6]=1 => b[c[1]-1] = b[1], c[1]=1
a[5]=4 => b[c[4]-1] = b[5], c[4]=5
a[4]... [c[3]-1] = b[3], c[3]=3
a[3]=1 => b[c[1]-1] = b[0], c[1]=0
a[2]=4 => b[c[4]-1] = b[4], c[4]=4
a[1]=6 => b[c[6]-1] = b[7], c[6]=7
a[0]=3 => b[c[3]-1] = b[2], c[3]=2

```
... j] ] = c[ a[j] ] + 1;

(3)for i=1 to k
     c[i] = c[i] + c[i-1];

   for j=n-1 to downto 0
     b[ c[a[j]]-1 ] = a[j];
     c[a[j]] = c[a[j]] - 1;
}
```

Today's Report

- In the previous slides, we prove that we need Ω(n log n) time for solving the sorting problem. On the other hand, counting sort runs in O(n) time. At a glance, it seems to be *contradiction*. But they are not conflict. Explain why.
- Deadline: 10am, Friday Morning