Introduction to Algorithms and Data Structures

7. Data structure (2) Binary Search Tree and its balancing

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Review:

- We have three combinations of “data structure”, “what to do” and “algorithm”.
- “What to do”: E.g., i-th data, search, add/insert/remove.

- Array: access in $O(1)$, search in $O(n)$
- Array in order: search in $O(\log n)$, but add/remove in $O(n)$
- Linked list: access in $O(n)$, but add/remove in $O(1)$
- Hash: easy to add and search
- Binary search tree: dynamic search

Dynamic search and data structure

- Sometimes, we would like to search in dynamic data, i.e., we add/remove data in the data set.
- Example: Document management in university
  - New students: add to list
  - Alumni: remove from list
  - When you get credit: search the list

Q. Good data structure?

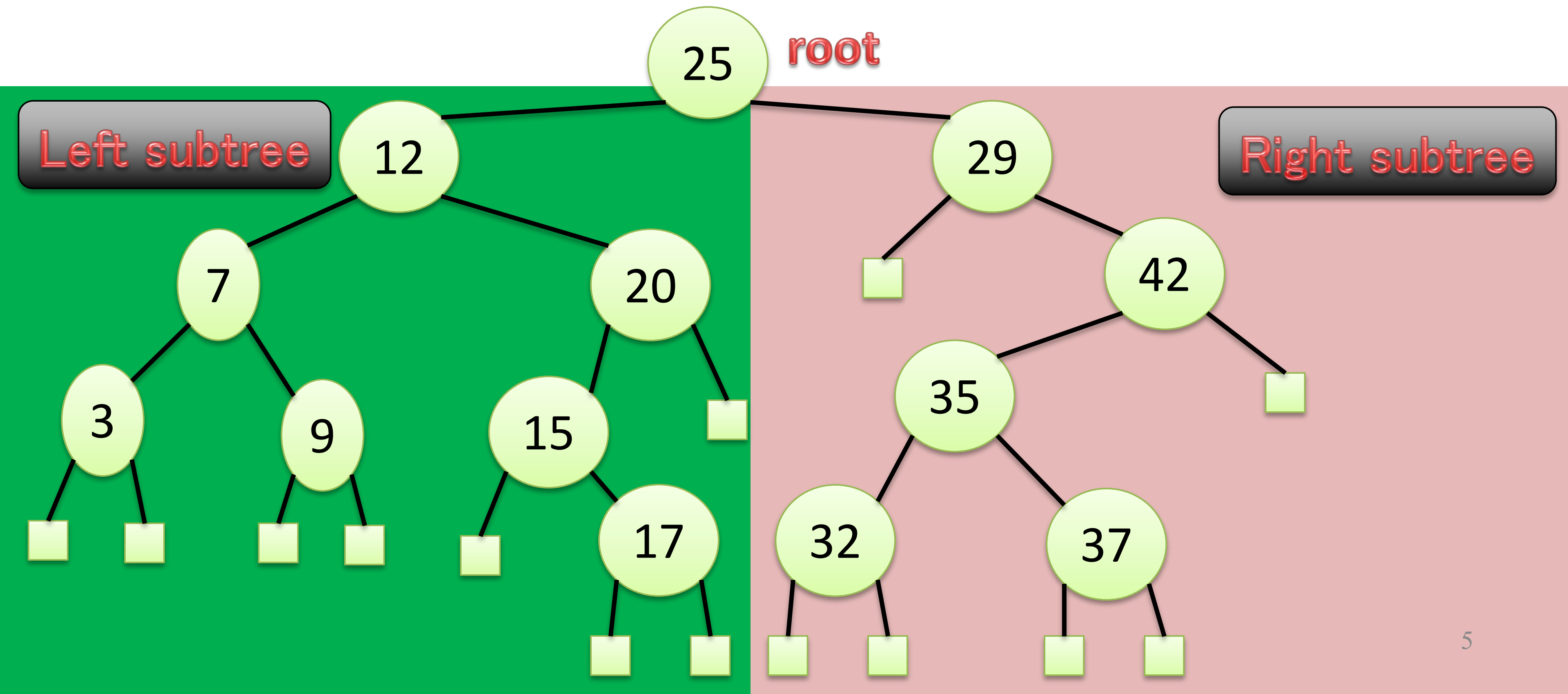
Naïve idea: array or linked list?

- Data in order:
  - Search: binary search in O(log n) time
  - Add and remove: O(n) time per data
- Data not in order:
  - Search and remove: O(n) time per data
  - Add: in O(1) time

Imagine: you have 10000 students, and you have 300 new students!

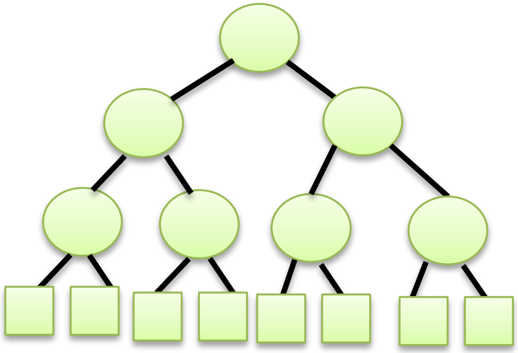
Better idea: binary search tree

- For every vertex v, we have the following;
  - Data in v ≧ any data in a vertex in left subtree
  - Data in v ≦ any data in a vertex in right subtree

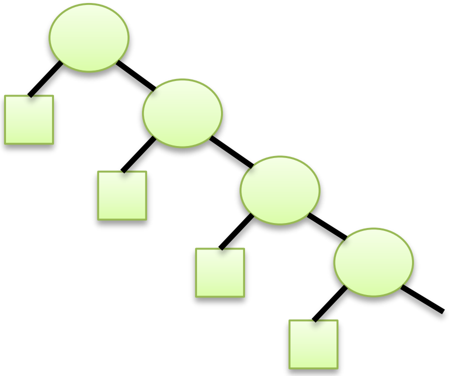


Better idea: binary search tree

- We construct binary search tree for a given data set; we learnt it can be updated in O(L) time, where L is the length of the route from a leaf to the root.



- When data is random:
  - Depth of the tree: O(log n)
  - Search, add, remove: O(log n) time.



- In the worst case:
  - Depth of the tree: n
  - When data is given in order, we have the worst case.
  - Search, add, remove: O(n) time…

Today: More binary search tree (BST)

1. Get maximum/minimum data (⇔ heap)
2. Enumerate all data in the tree（⇔ array）
3. “Good” and “bad” structure?
4. How can we fix bad to good?

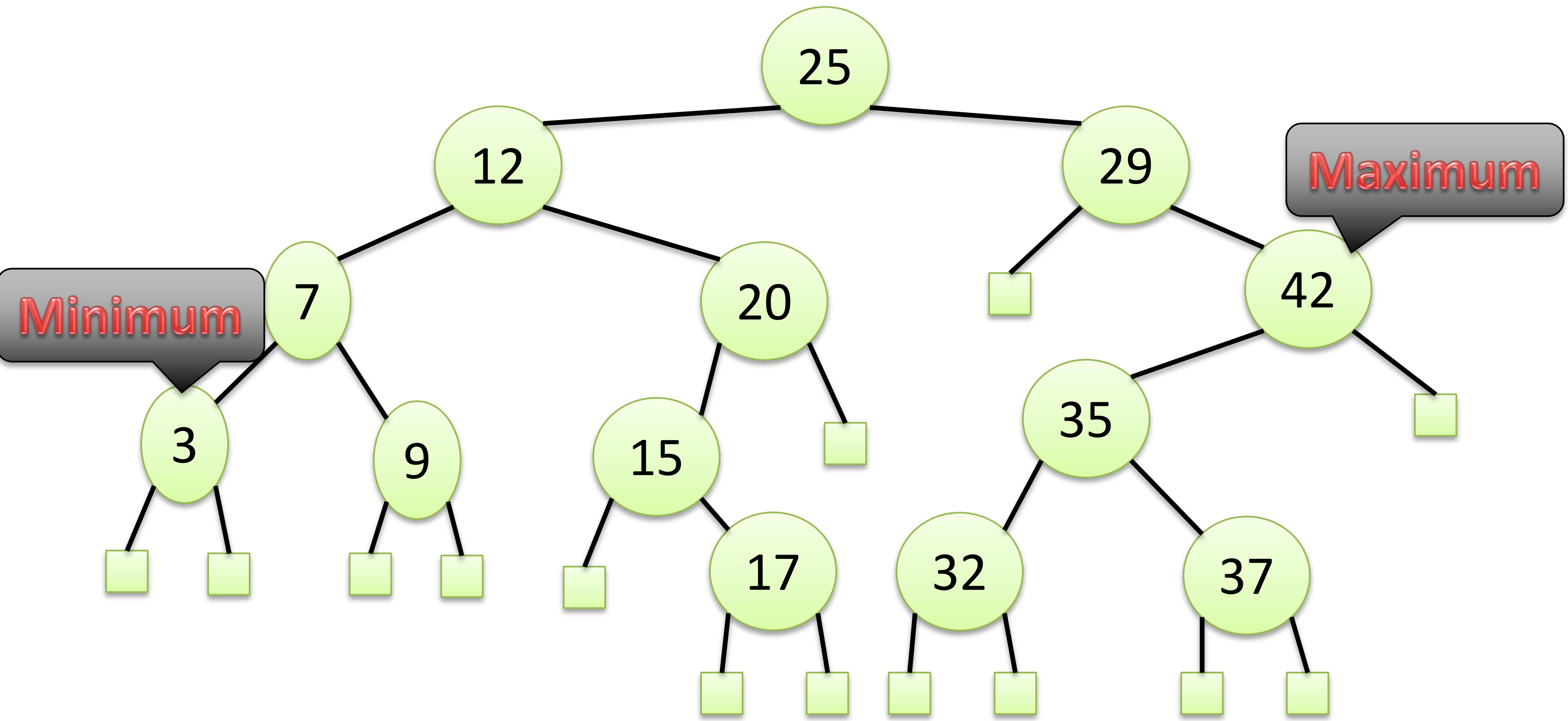
1. Max/min data in BST

- Properties of a BST
  - All left descendants have smaller values
  - All right descendants have larger values
- Using the properties…
  - Minimum: the leftmost lowest descendant from the root
  - Maximum: the rightmost lowest descendant from the root

- Tips: It is easy to remove the minimum/maximum node (since it has at most one child)

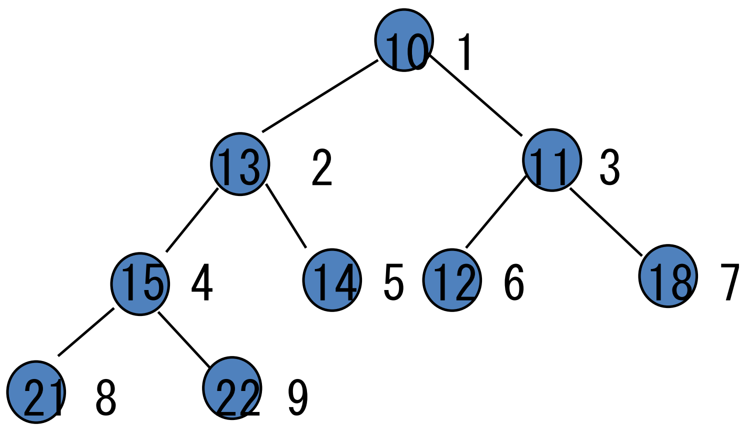
1. Max/min data in BST (Example)

(consider remove them also)



[Review]

How about heap?



1. Assign 1 to the root.
2. For a node of number i, assign 2 × i to the left child and assign 2 × i+1 to the right child.
3. No nodes assigned by the number greater than n.
4. For each edge, parent stores data smaller than one in child.

We can use an array, instead of linked list!

1	2	3	4	5	6	7	8	9
10	13	11	15	14	12	18	21	22

- It is easy to obtain the minimum one (at root)
- However, maximum one is not easy in the tree/array

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We have three ways of enumeration

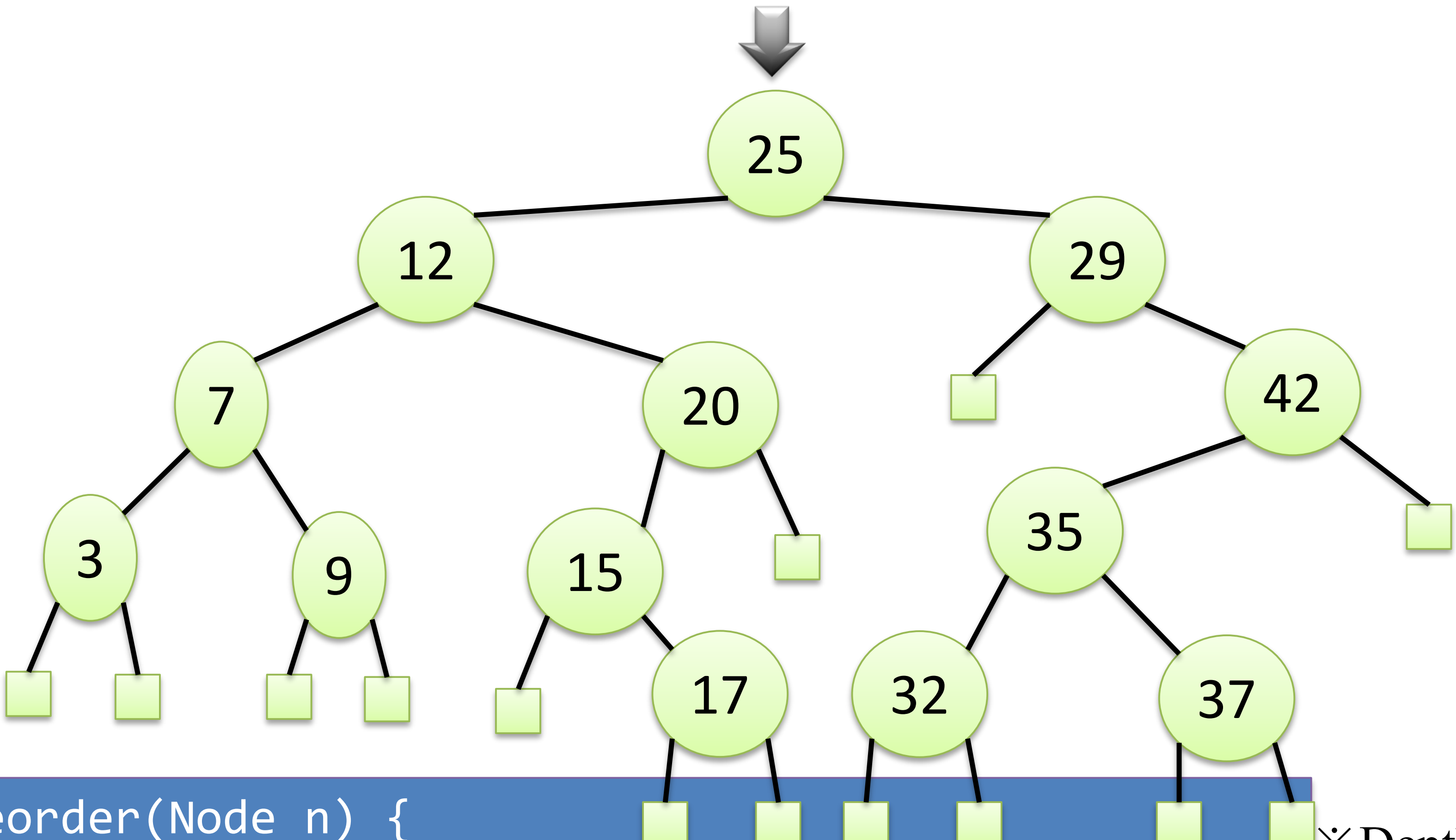
(general traverse ways of a binary tree)

- Preorder:
  Data in the current node → left subtree → right subtree
- Inorder:
  left subtree → Data in the current node → right subtree
- Postorder:
  left subtree → right subtree → Data in the current node

※It is easy to enumerate all data in array or linked list

How to traverse binary tree: preorder

Data in node → left subtree → right subtree



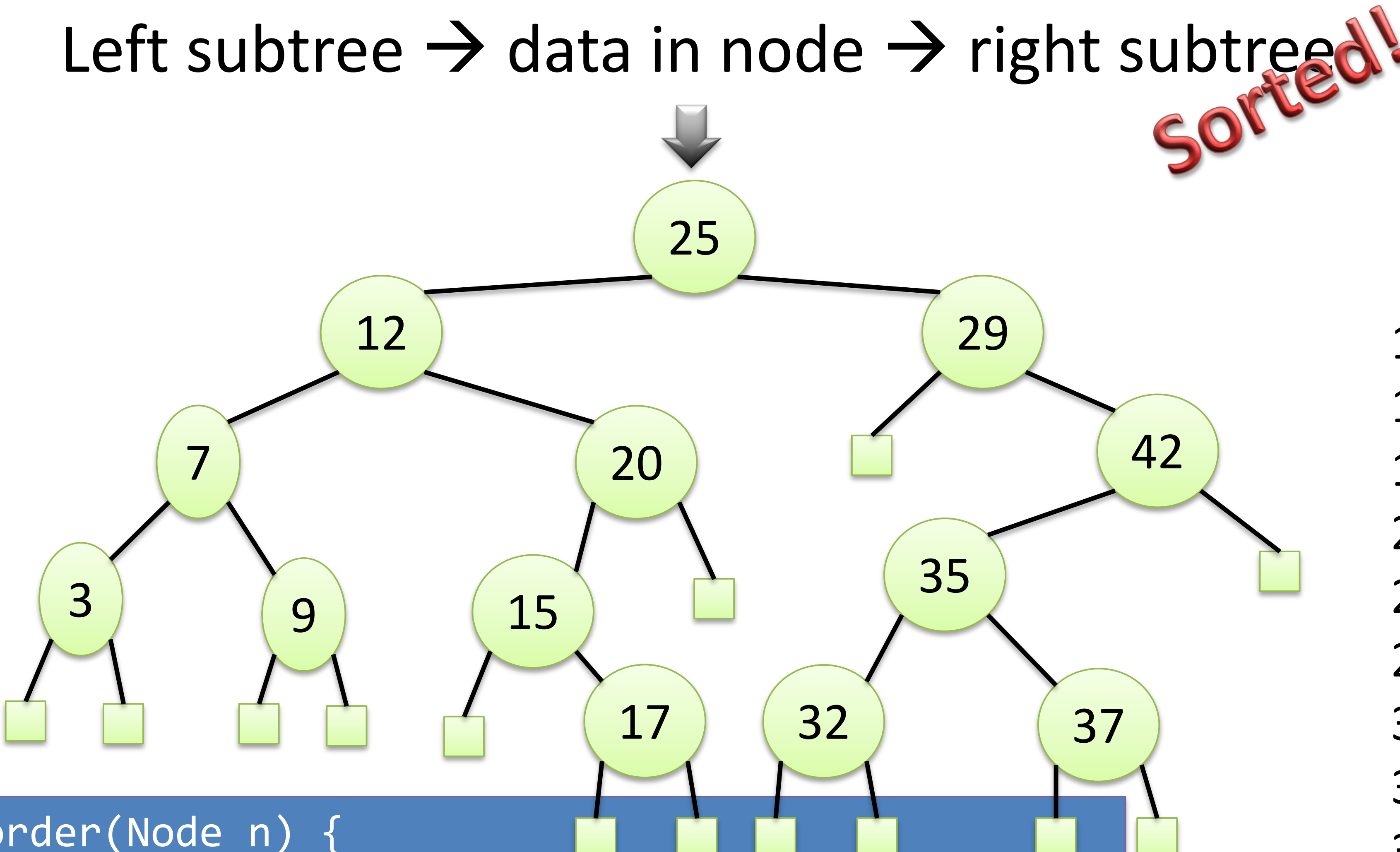
25
12
7
3
9
20
15
17
29
42
35
32
37

```
preorder(Node n) {
    if (n==null) return;
    visit(n); preorder(n.lson); preorder(n.rson);
}
```

※Depth first manner

How to traverse binary tree: inorder

Left subtree → data in node → right subtree

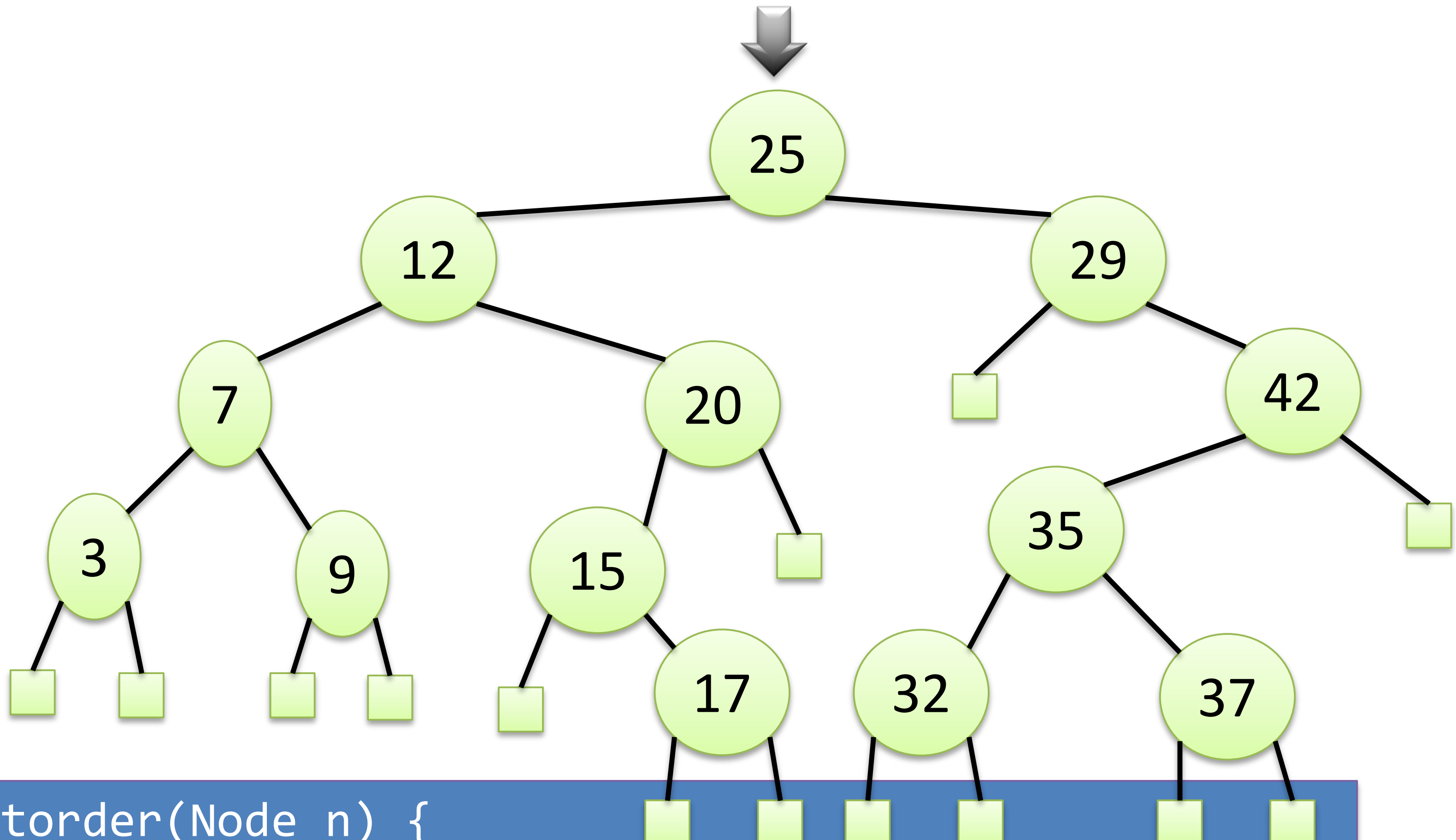


3
7
9
12
15
17
20
25
29
32
35
37
42

```
inorder(Node n) {
    if (n==null) return;
    inorder(n.lson); visit(n); inorder(n.rson);
}
```

How to traverse binary tree: postorder

Left subtree → right subtree → data in node



3
9
7
17
15
20
12
32
37
35
42
29
25

```
postorder(Node n) {
    if (n==null) return;
    postorder(n.lson); postorder(n.rson); visit(n);
}
```

Example of code

```
public class I111_08_p22{
    public static void Main(){
        Node n3 = new Node (3, null, null);
        Node n9 = new Node (9, null, null);
        Node n7 = new Node (7, n3, n9);
        Node n17 = new Node (17, null, null);
        Node n15 = new Node (15, null, n17);
        Node n20 = new Node (20, n15, null);
        Node n12 = new Node (12, n7, n20);
        Node n32 = new Node (32, null, null);
        Node n37 = new Node (37, null, null);
        Node n35 = new Node (35, n32, n37);
        Node n42 = new Node (42, n35, null);
        Node n29 = new Node (29, null, n42);
        Node n25 = new Node (25, n12, n29);

        inorder(n25);
    }

    static void inorder(Node n) {
        if (n==null) return;
        inorder(n.lson);
        visit(n);
        inorder(n.rson);
    }

    static void visit(Node n) {
        System.Console.Write(n.data+" ");
    }
}
```

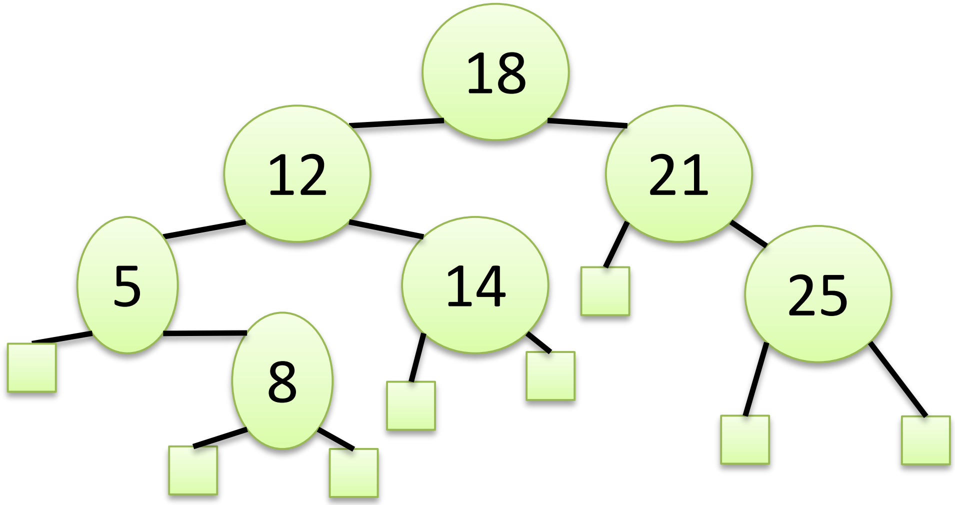
Easy to modify to pre, post

output

```
public class Node {
    public int data;
    public Node lson;
    public Node rson;
    public Node (int i, Node ls, Node rs) {
        data = i;
        lson = ls;
        rson = rs;
    }
}
```

Small exercise

- Make a small binary search tree (around 10 nodes)
- Find the maximum and minimum data
- Remove the root node
- Enumerate data in preorder, inorder, and postorder



Today: More binary search tree (BST)

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2. Enumerate all data in the tree（⇔ array）
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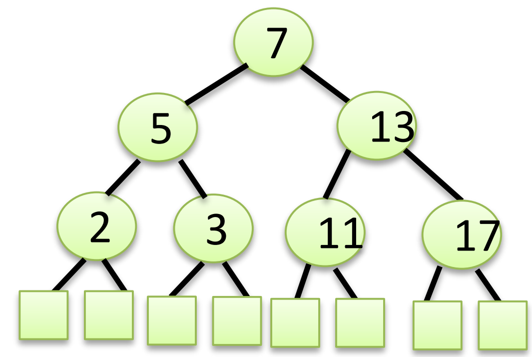
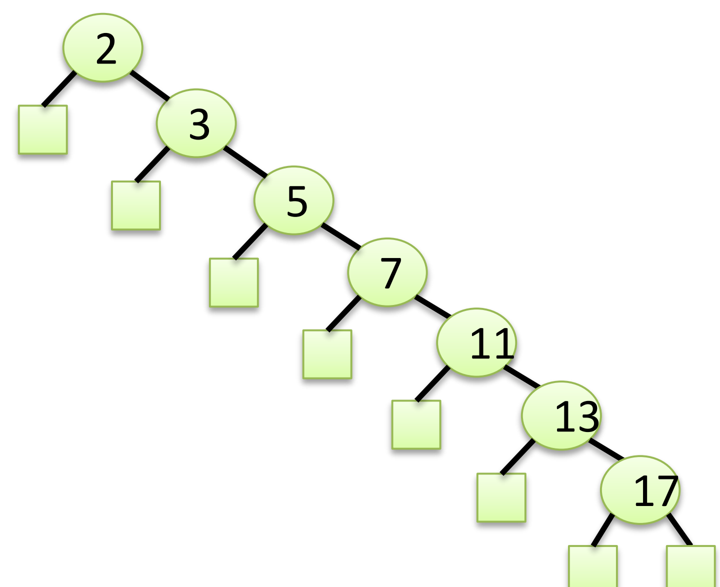
4. How can we fix bad to good?

Efficiency of BST

- Best case: O(log n)
  - Each of n data is kept in BST of depth $\log_2 n$



- Worst case: O(n)
  - If we put in increasing order→ we have depth n



- "Random order" is also interesting topic, but we make it of depth O(log n) in any case.

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Nice idea: (Self-)Balanced Binary Search Tree

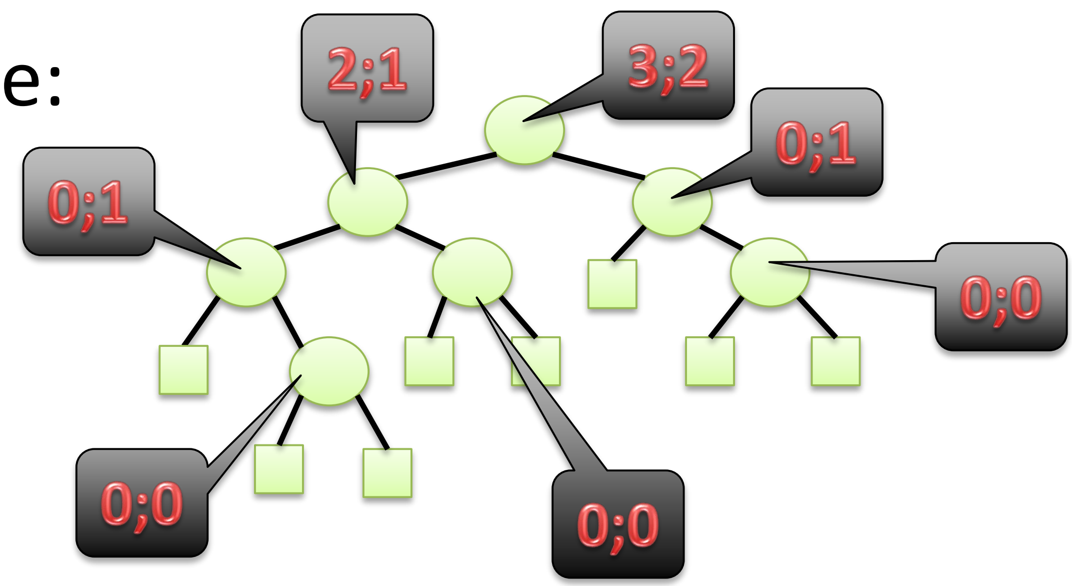
- There are some algorithms that maintain to take balance of tree in depth $O(\log n)$.
  - e.g., AVL tree, 2-3 tree, 2-color tree (red-black tree)

Georgy M. Adelson-Velsky
(1922−2014)

Evgenii M. Landis
(1921−1997)

AVL tree [G.M. Adelson-Velskii and E.M. Landis '62]

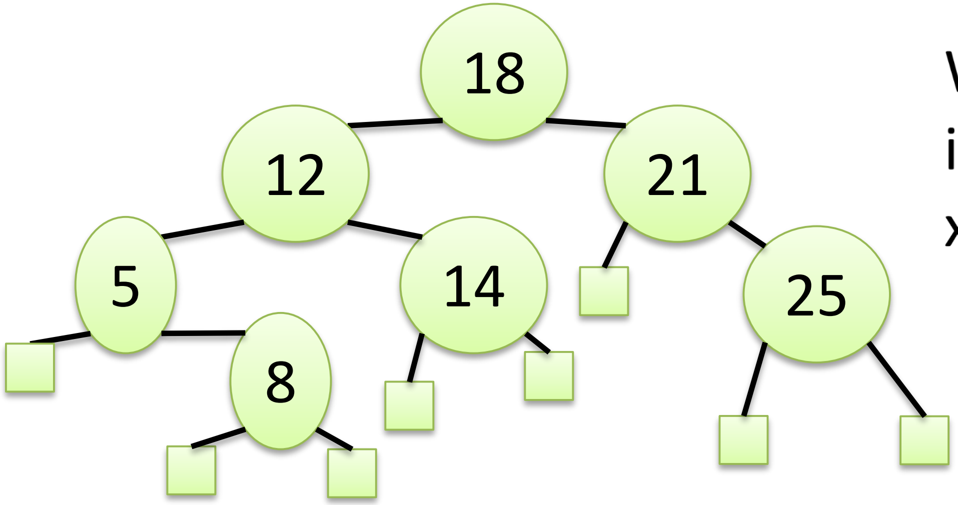
- Property (or assertion): at each vertex, the depth of left subtree and right subtree differs at most 1.
- Example:



AVL tree: Insertion of data

- Find a leaf v for a new data x
- Store data x into v (v is not a leaf any more)
- Check the change of balance by insertion of x
- From v to the root, check the balance at each vertex, and rebalance (rotation) if necessary.

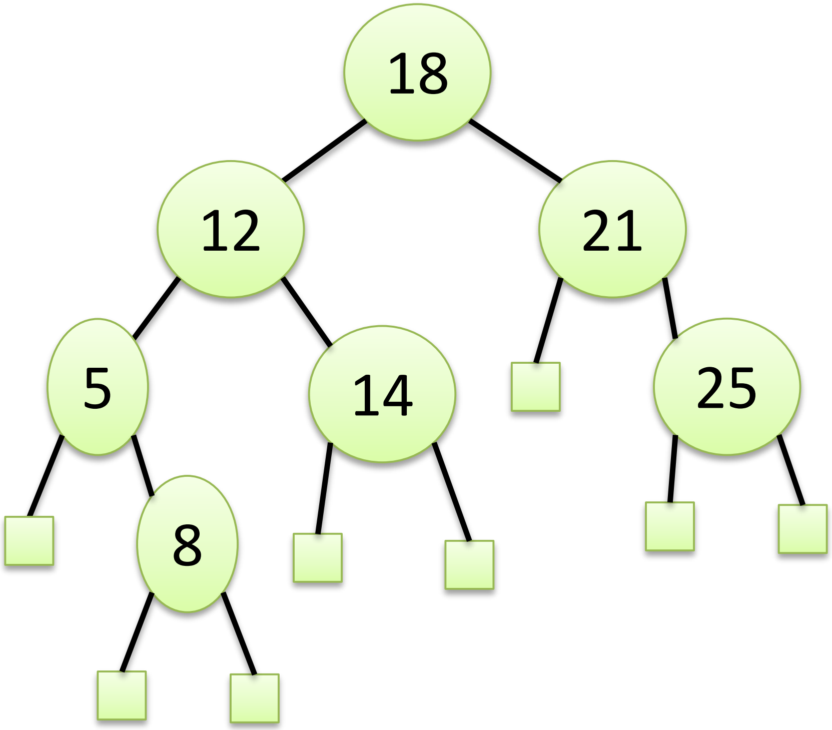
We have nothing to do up to here



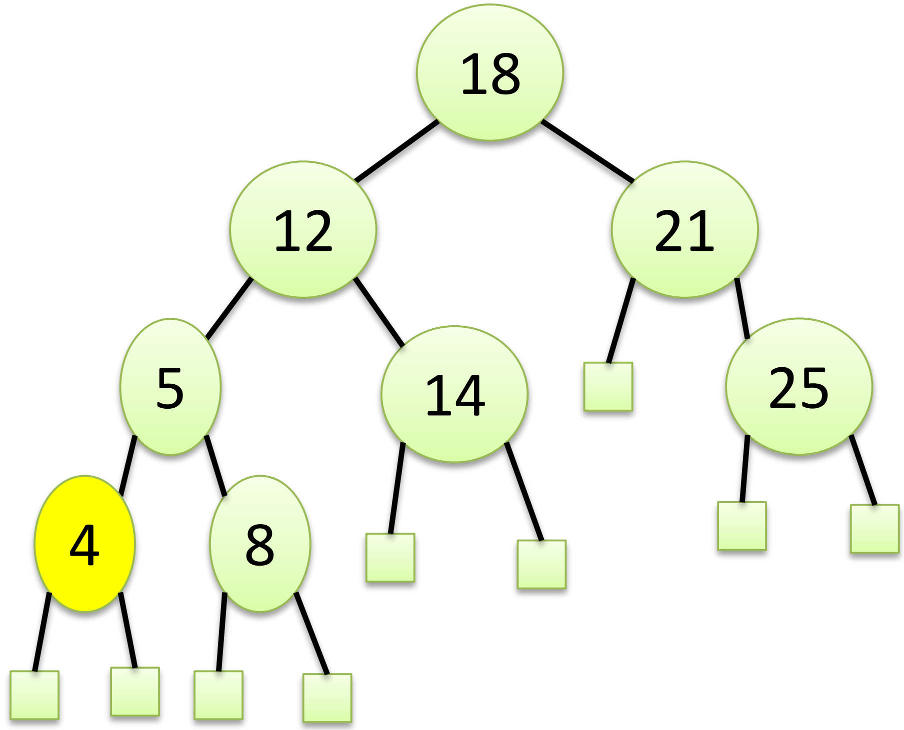
What happens if you insert x=4? How about x=10, x=20, x=23?

AVL tree: Insertion of data
Insert x=4

before



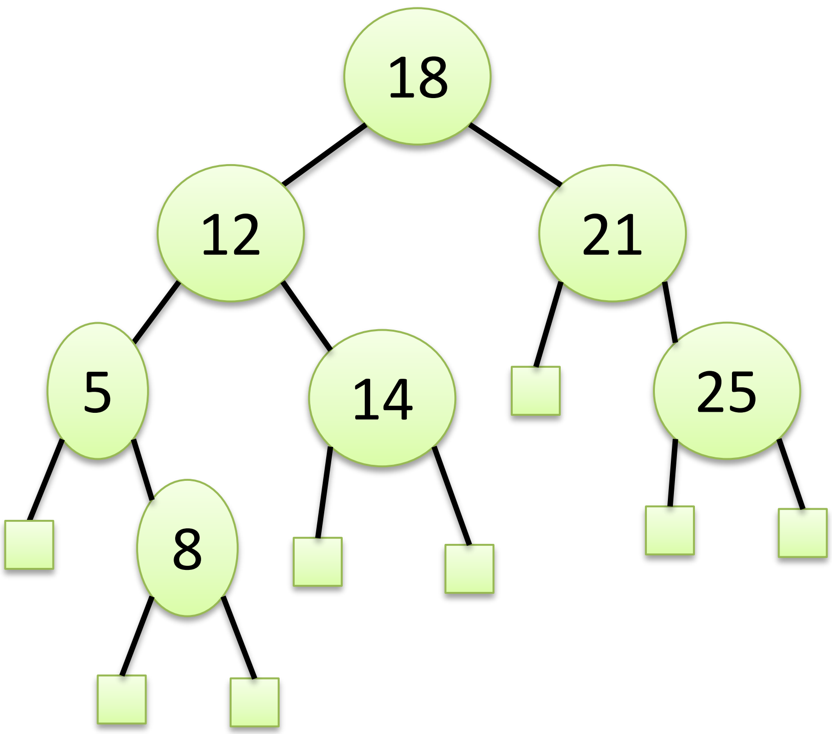
after



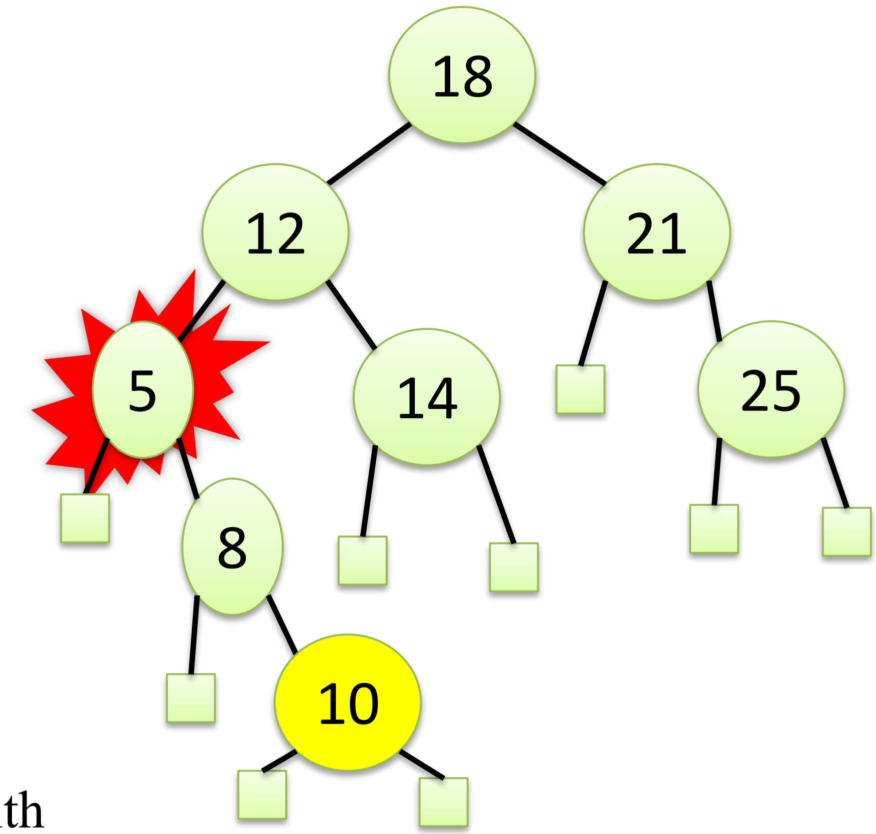
Balance: OK

AVL tree: Insertion of data
Insert x=10

before



after

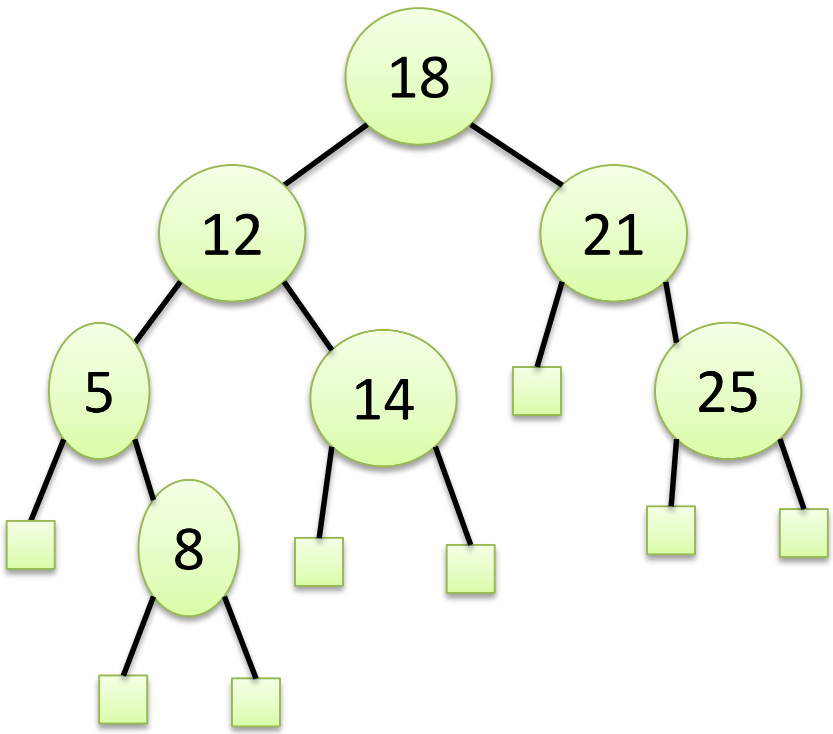


※We also have unbalanced at 12 and 18 with 1:3 and 2:4, resp, but we first handle the deepest point

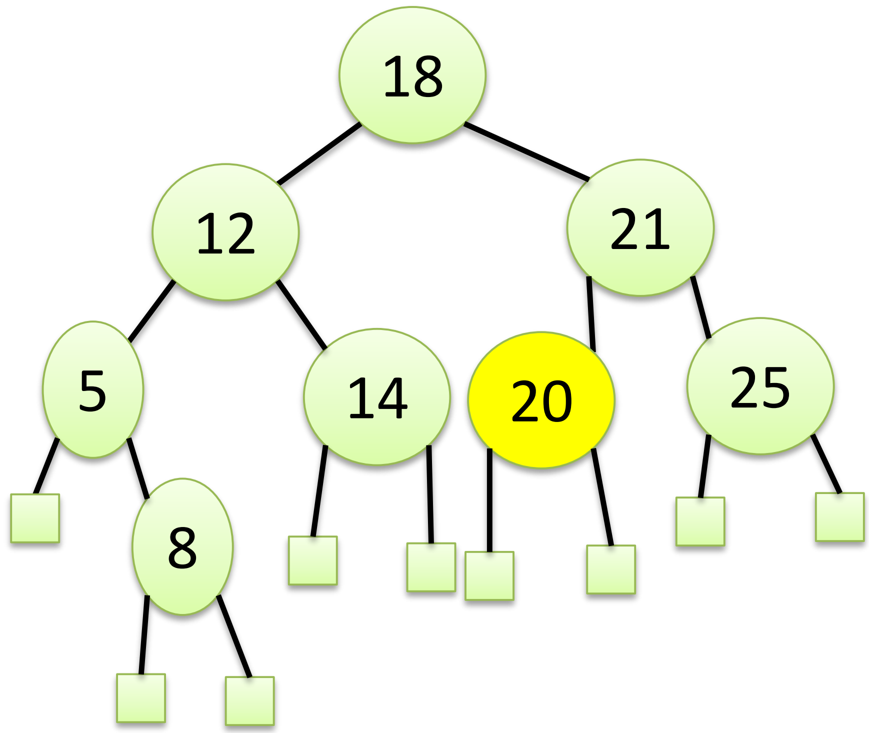
0;2@vertex 5

AVL tree: Insertion of data
Insert x=20

before



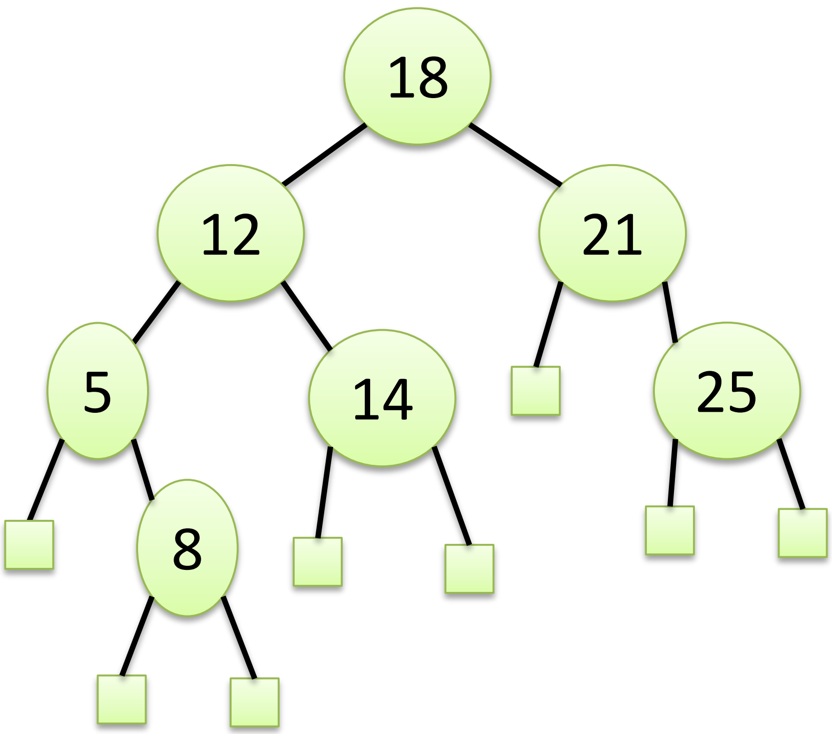
after



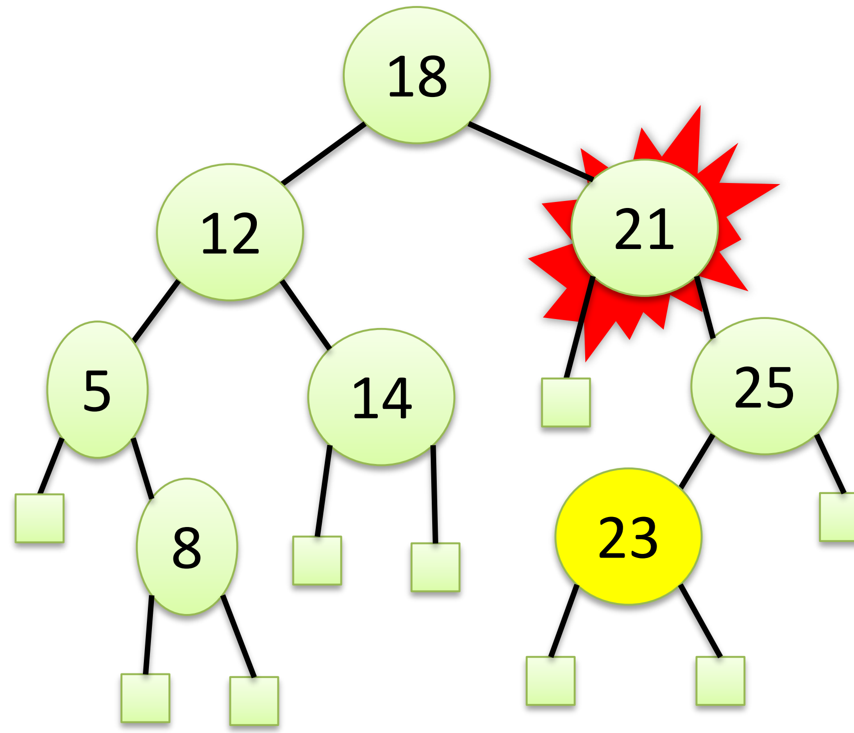
Balance: OK

AVL tree: Insertion of data
Insert x=23

before



after



0;2@vertex 21

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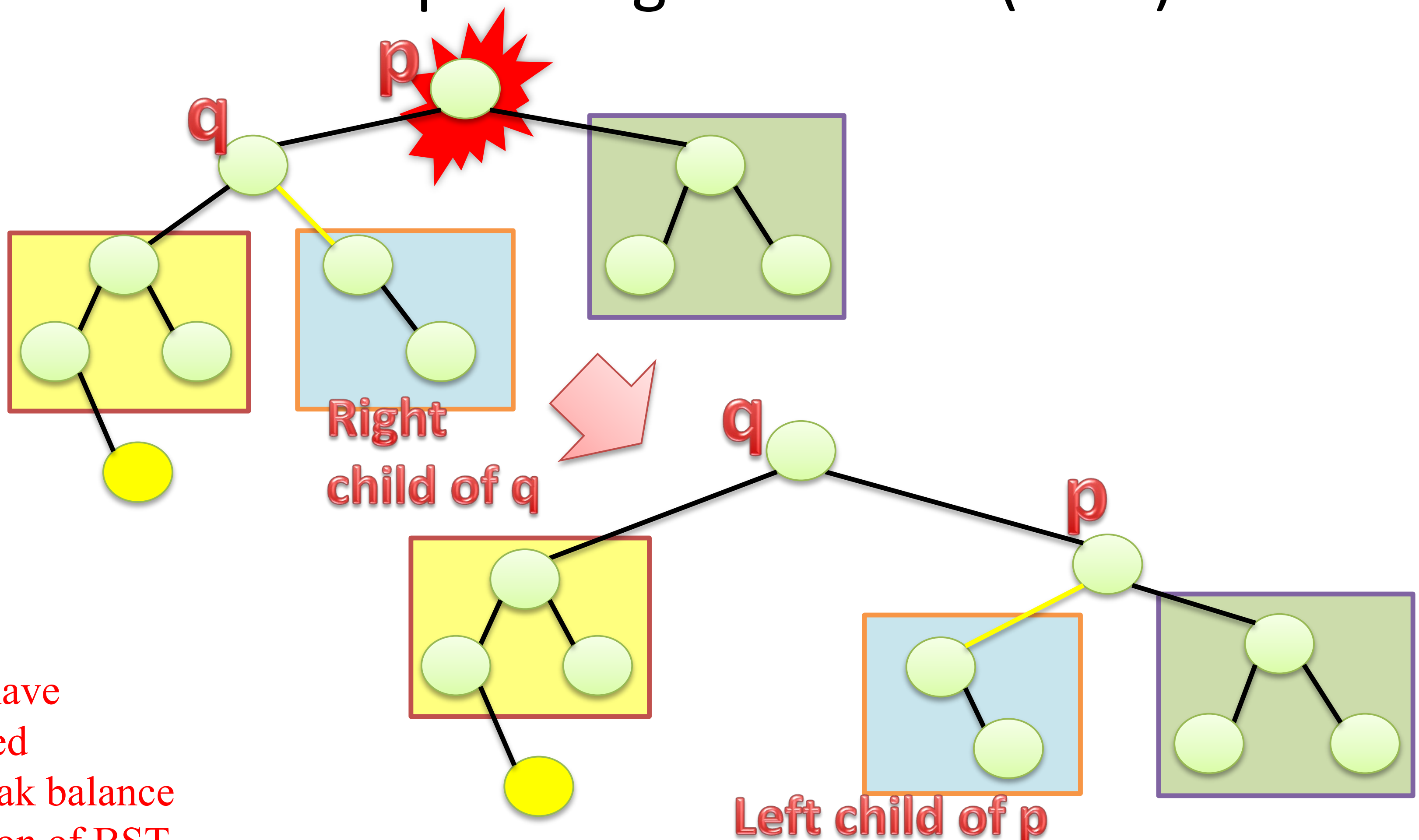
AVL tree: Rebalance by rotations

- If you insert/remove data, the BST can get unbalanced.
- "Rotate" tree vertices to make the difference up to 1:
  - Rotation LL
  - Rotation RR
  - Double rotation LR
  - Double rotation RL

Rebalance of AVL-tree by rotation: Rotation LL

- Lift up left subtree (yellow) if too deep
  we have to transplant right subtree (blue)

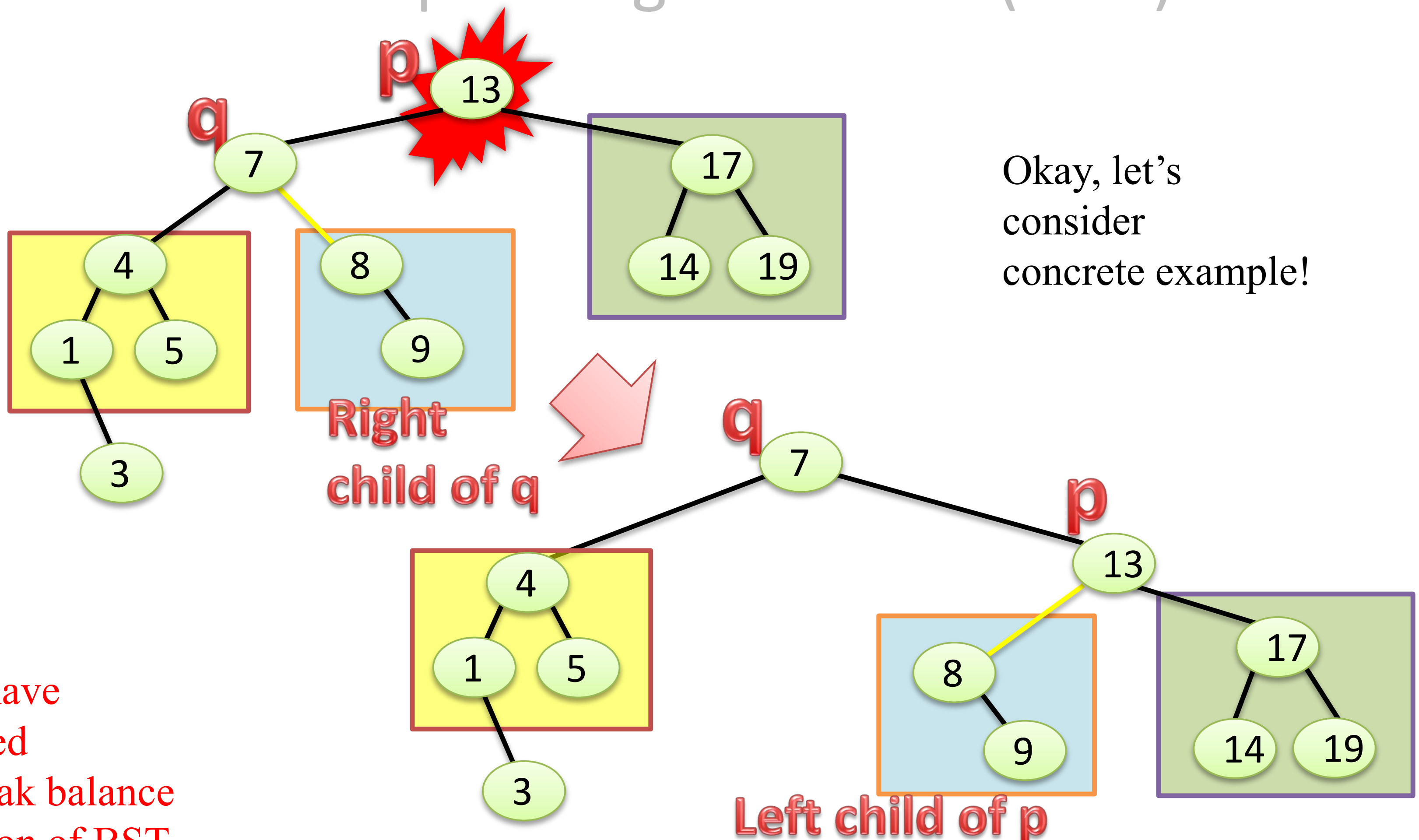


Now we have
- balanced
- not break balance
- condition of BST

Rebalance of AVL-tree by rotation: Rotation LL

- Lift up left subtree (yellow) if too deep

  we have to transplant right subtree (blue)



Okay, let's consider concrete example!

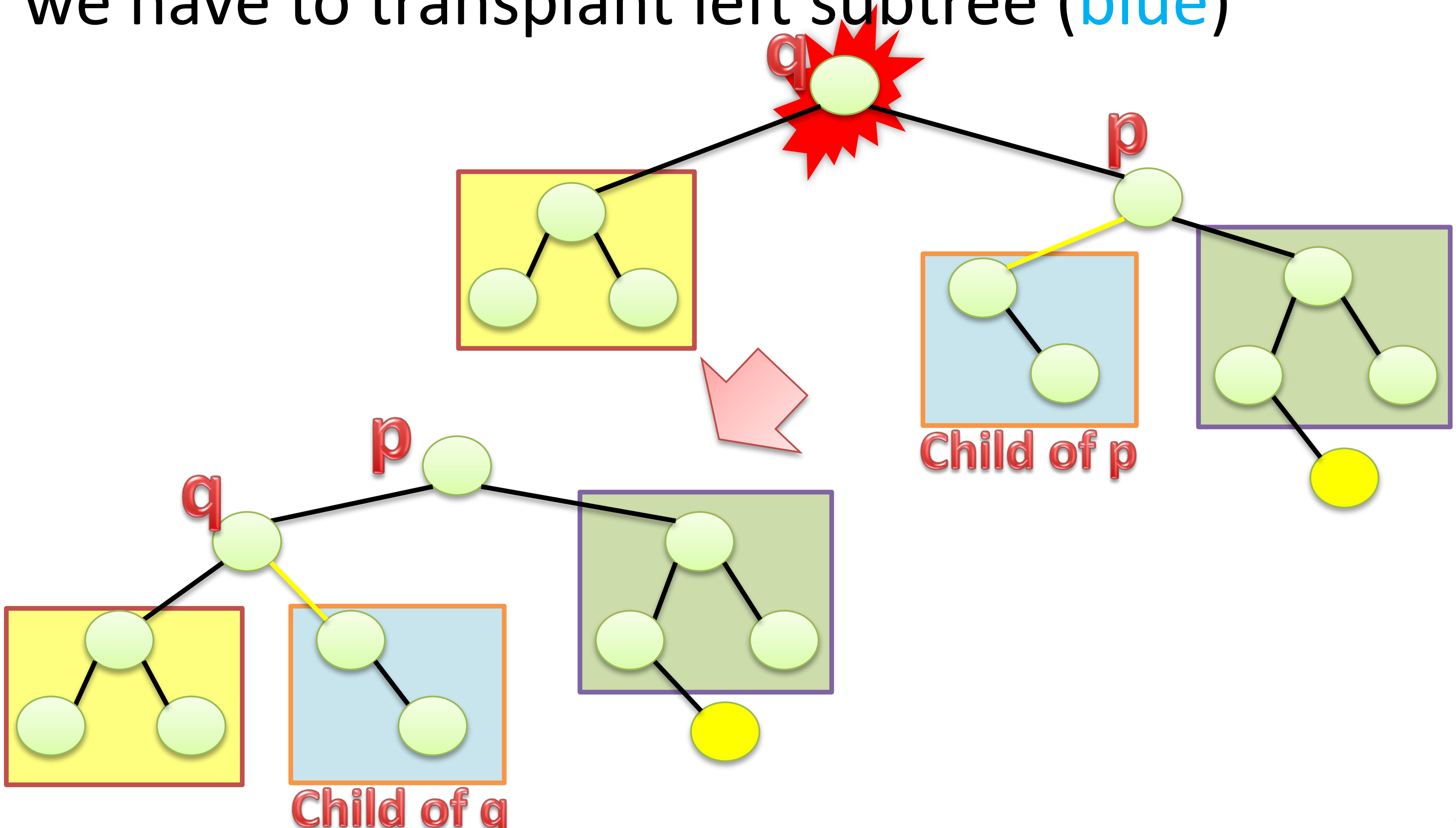
Now we have

- balanced
- not break balance
- condition of BST

Rebalance of AVL-tree by rotation: Rotation RR (just mirror image of LL)

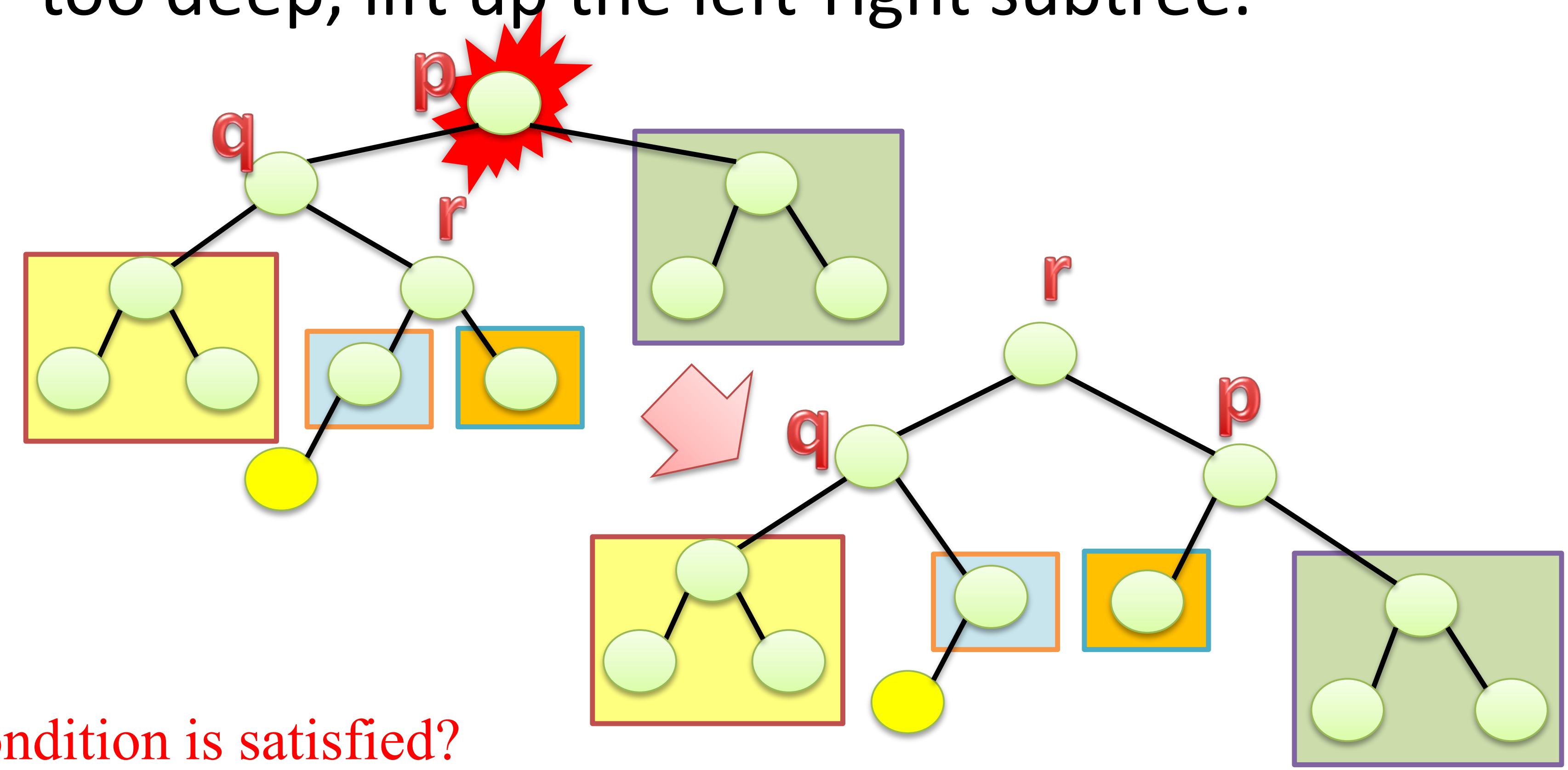
- Lift up right subtree (green) if too deep

  we have to transplant left subtree (blue)



AVL tree: Rebalance by rotation: Double rotation LR

- When right subtree of left subtree becomes too deep, lift up the left-right subtree.

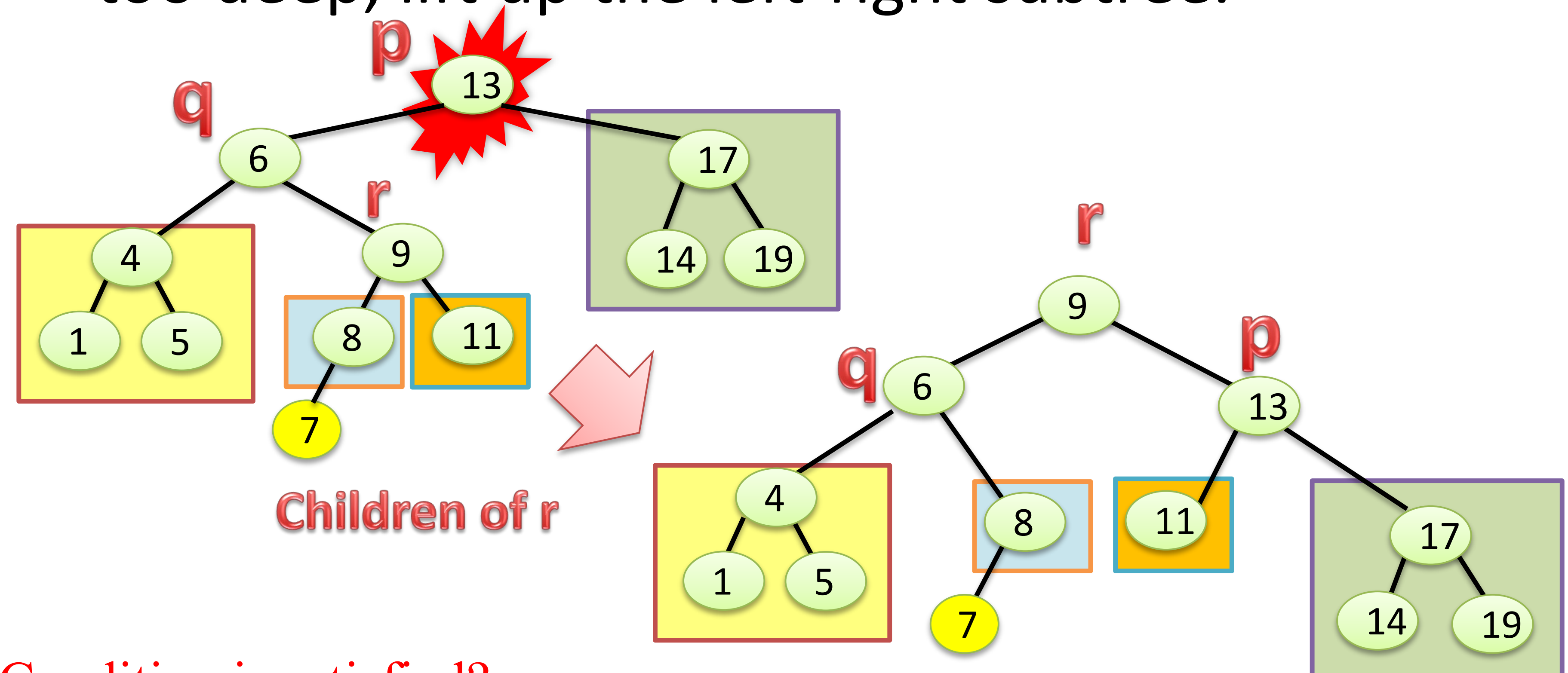


※Condition is satisfied?

※Why rotation LL does not work?

AVL tree: Rebalance by rotation: Double rotation LR

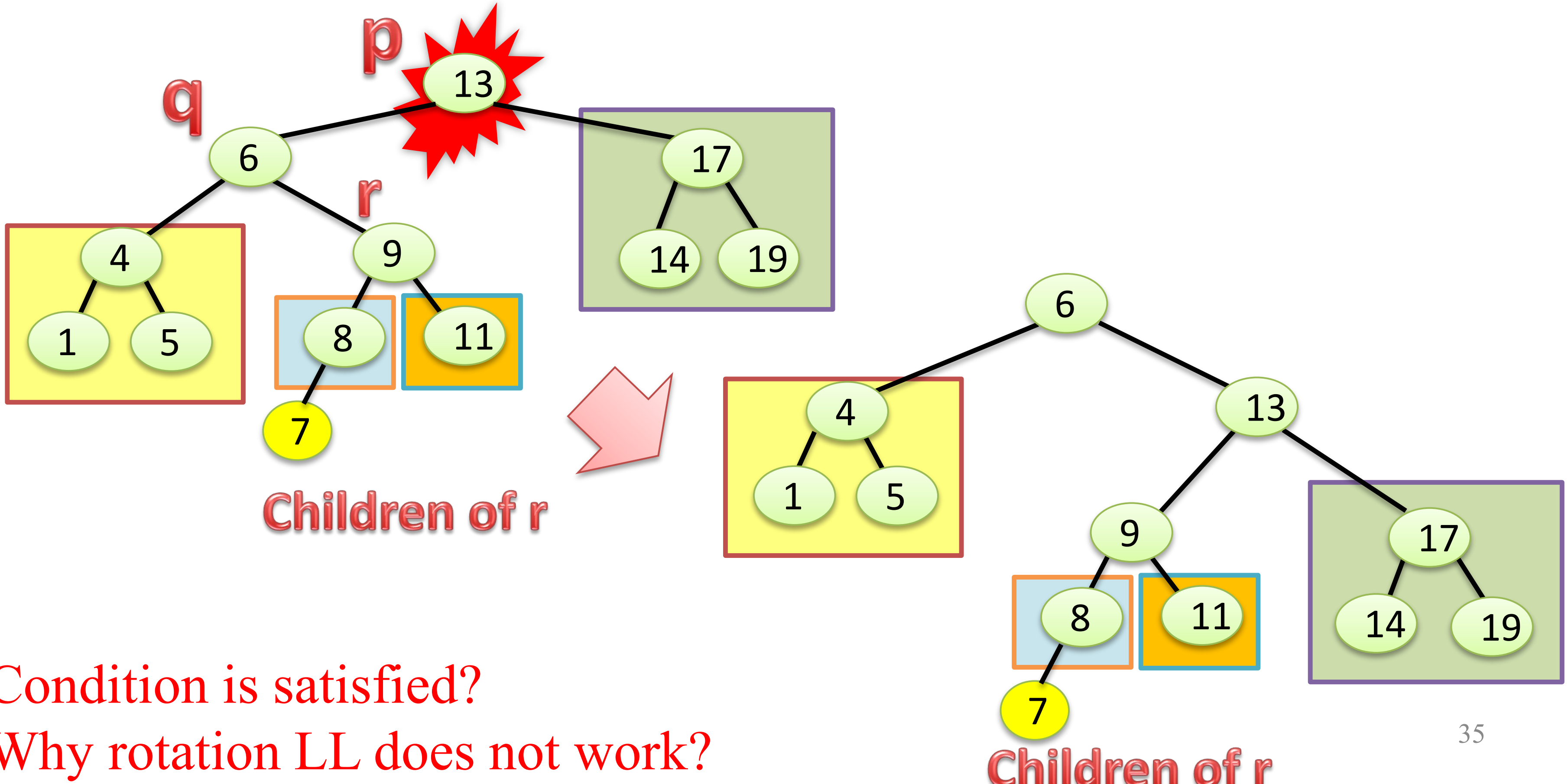
- When right subtree of left subtree becomes too deep, lift up the left-right subtree.



※Condition is satisfied?

※Why rotation LL does not work?

(If you apply rotation LL)

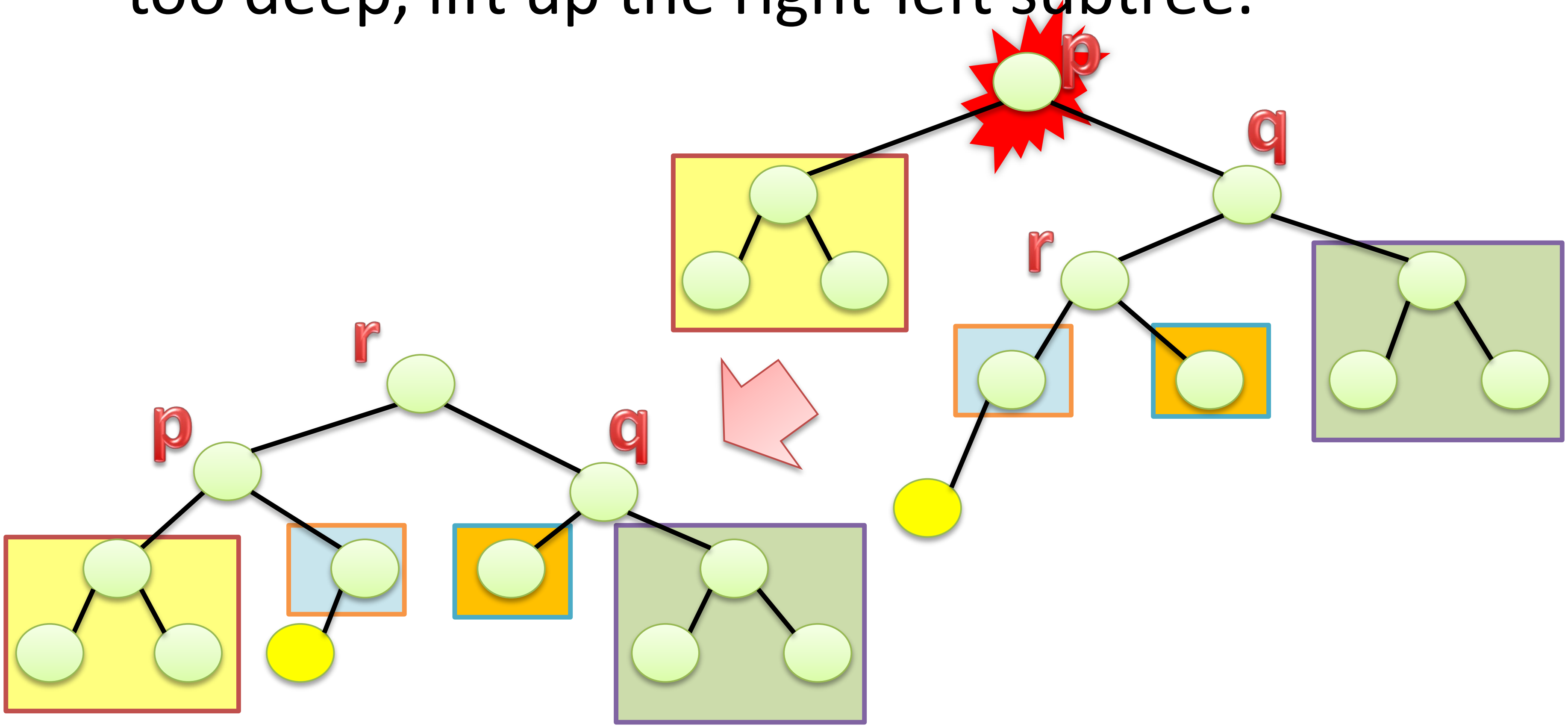


※Condition is satisfied?

※Why rotation LL does not work?

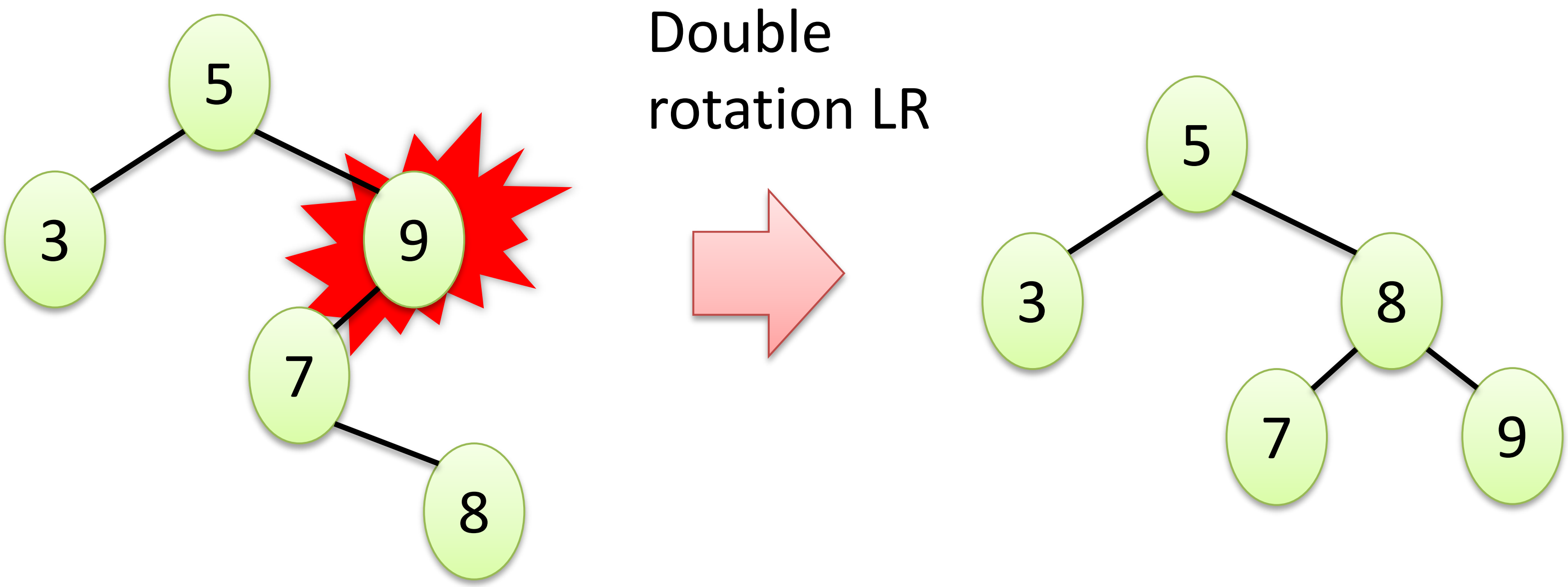
AVL tree: Rebalance by rotation: Double rotation RL (just mirror image of LR)

- When left subtree of right subtree becomes too deep, lift up the right-left subtree.



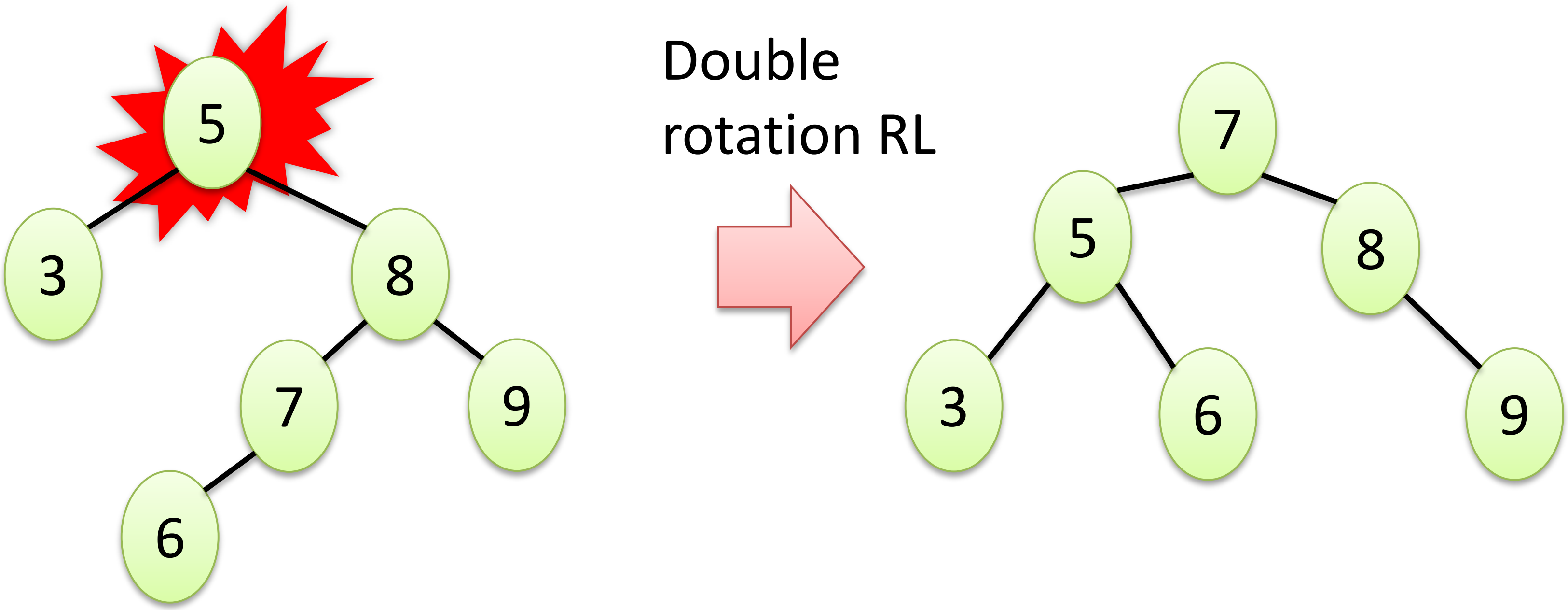
AVL tree: Example

- Insertion of 8



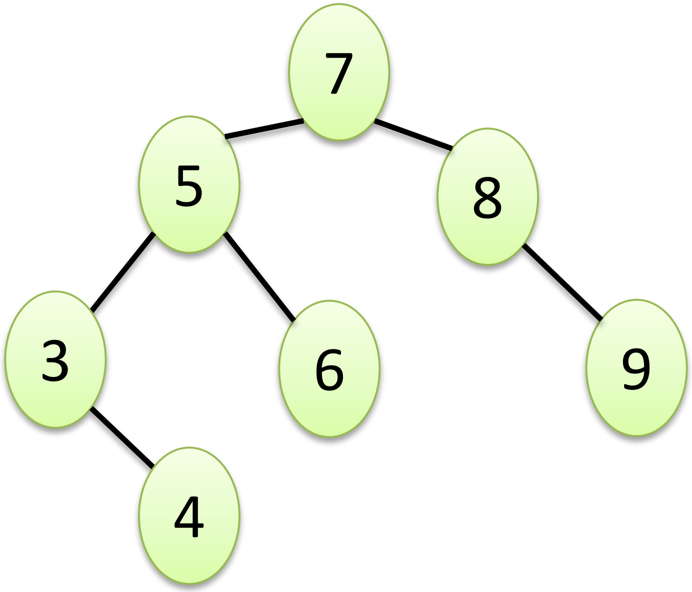
AVL tree: Example

- Insertion of 6



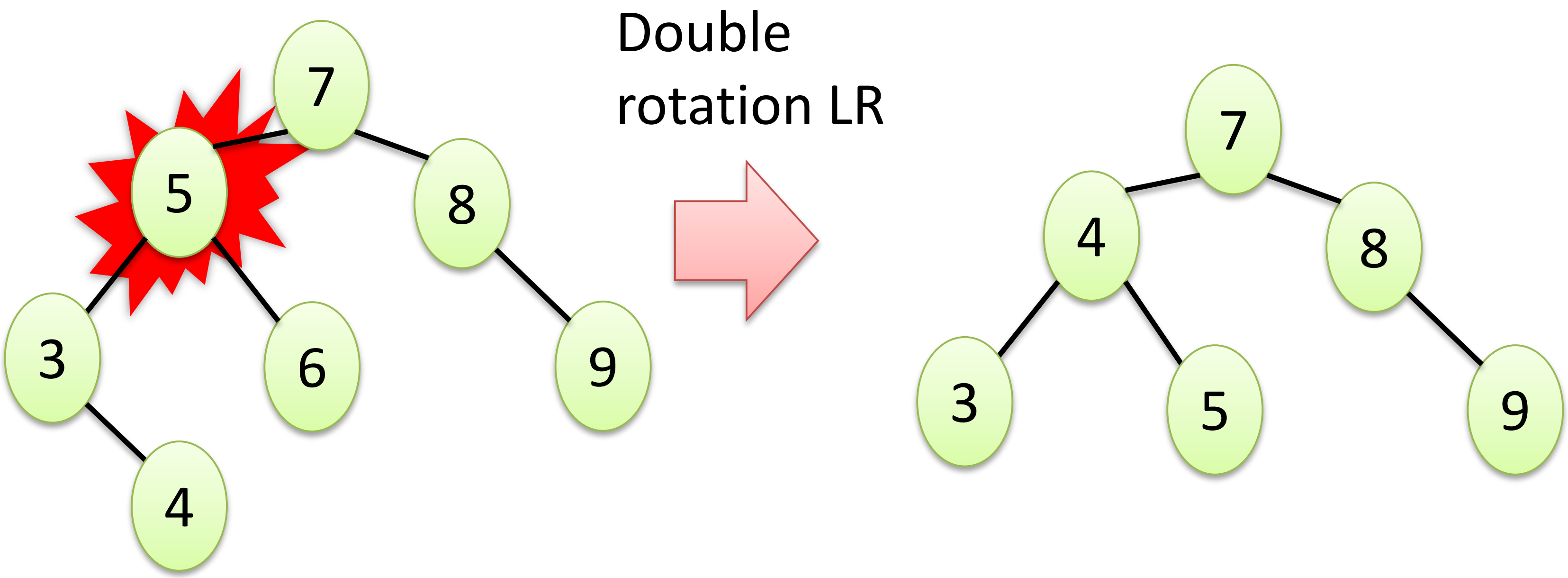
AVL tree: Example

- Insertion of 4 (balance is okay)



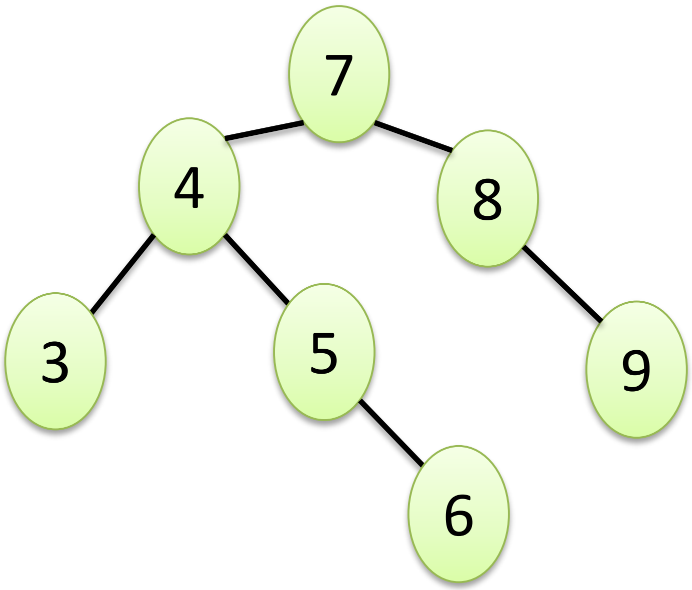
AVL tree: Example

- Deletion of 6



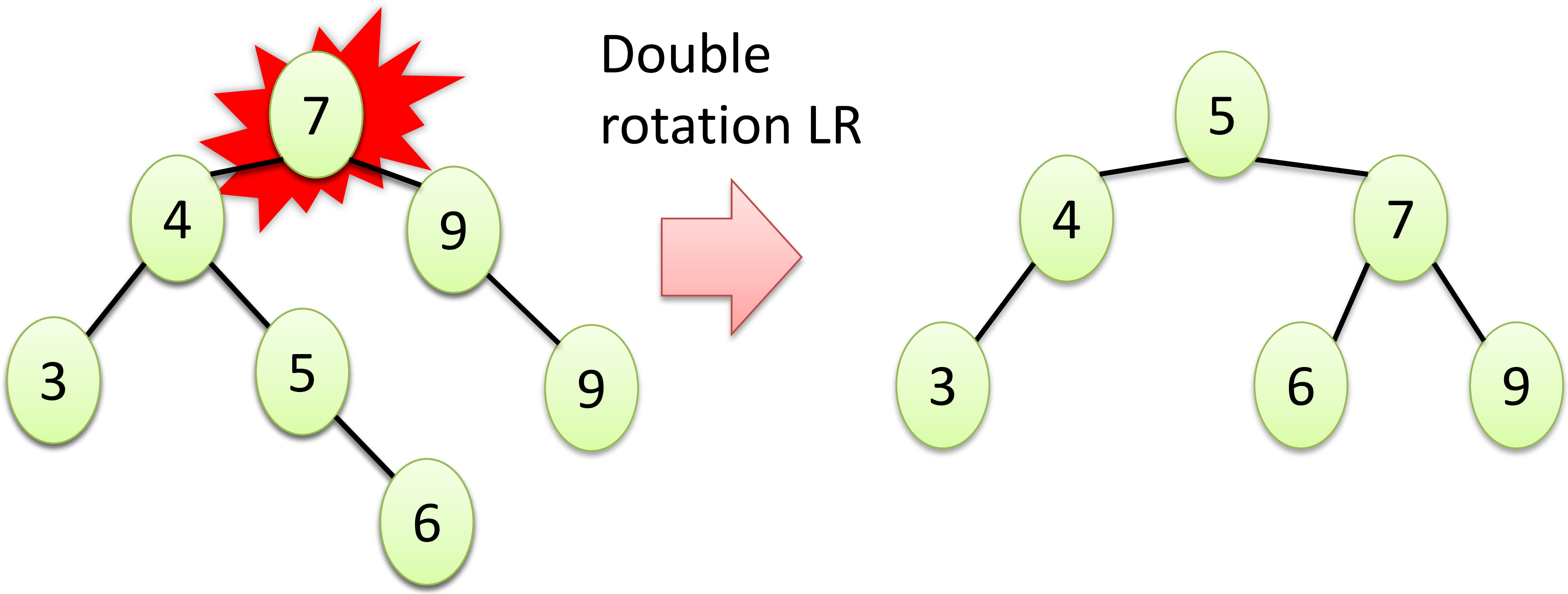
AVL tree: Example

- Insertion of 6 (balance is okay)



AVL tree: Example

- Deletion of 8



Time complexity of balanced binary search tree

- Search: $O(\log n)$ time
- Insertion/Deletion: $O(\log n)$ time
  - $O(\log n)$ rotations
  - Each rotation takes constant time

- In total, on a balanced binary search tree, every operation can be done in $O(\log n)$ time.

  (n is the number of data in the tree)