Introduction to Algorithms and Data Structures

Lecture 14: Graph Algorithms (1)

Breadth-first search and Depth-first search

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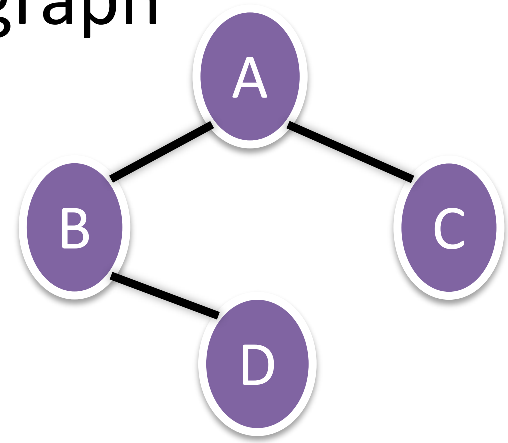
Search in Graph

- How can we check all vertices in a graph systematically, and solve some problem?
  - e.g., Do you have a path from A to D?
- Two major (efficient) algorithms:
  - Breadth First Search: A -> B -> C -> D
    it starts from a vertex v, and visit all (reachable) vertices from the vertices closer to v.
  - Depth First Search: A -> B -> D -> C
    it starts from a vertex v, and visit every reachable vertex from the current vertex, and back to the last vertex which has unvisited neighbor.



BFS (Breadth-First Search)

- For a graph $G=(V,E)$ and any start point $s \in V$, all reachable vertices from s will be visited from s in order of distance from s.
- Outline of method: color all vertices by white, gray, or black as follows;
  - White: Unvisited vertex
  - Gray: It is visited, but it has unvisited neighbors
  - Black: It is already visited, and all neighbors are also visited

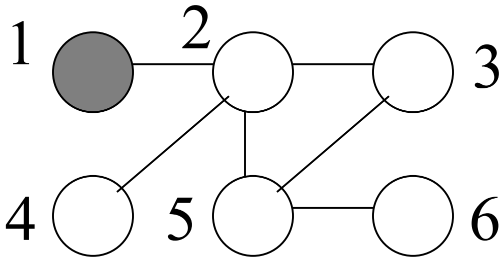
  - Search is completed when all vertices got black
  - Color of each vertex is changed as white→gray→black

BFS (Breadth-First Search): Program code

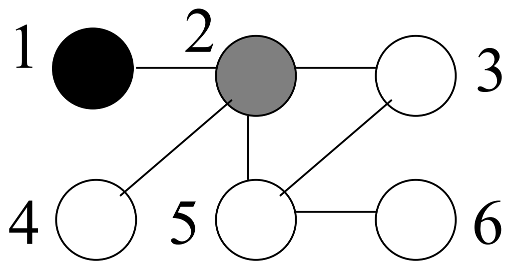
```
BFS(V,E,s){
  for v∈V do toWhite(v); endfor
  toGray(s);
  Q={s};
  while( Q!={} ){
    u=pop(Q); // Q → Q’ where Q={u}UQ’
    for v∈{v∈V|(v,u)∈E}
      if isWhite(v) then
        toGray(v); push(Q,v);
      endif
    endfor
    toBlack(u);
  }
}
```

Queue is the best data structure for this purpose!

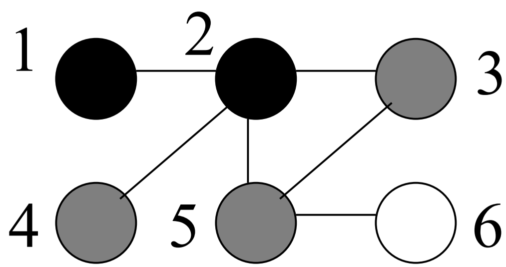
BFS (Breadth-First Search): Example



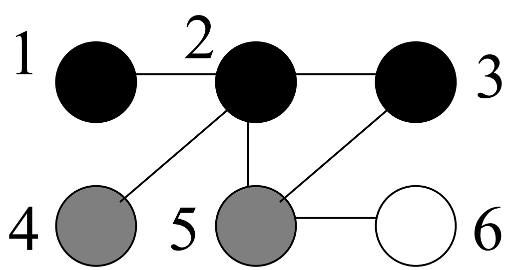
Q={1}



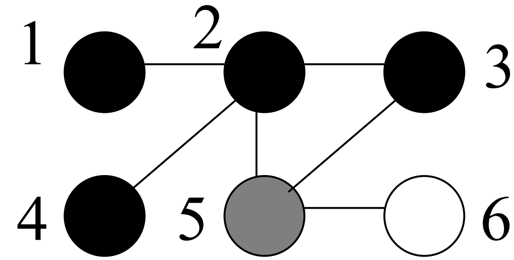
u=1,
visit 2
Q={2}
black 1



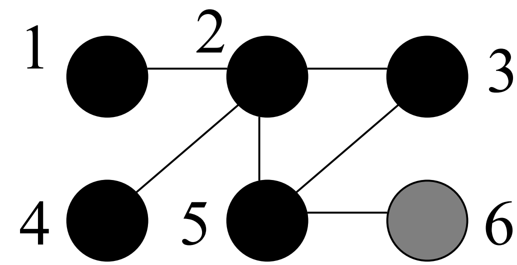
u=2,
visit 3,4,5
Q={3,4,5}
black 2



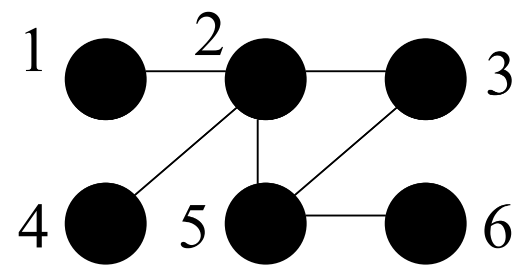
u=3,
visit null
Q={4,5}
black 3



u=4,
visit null
Q={5}
black 4



u=5,
visit 6
Q={6}
black 5



u=6,
visit null
Q={}
black 6

BFS: Time complexity

Time complexity is not easy from program...

Consider from

<u>the viewpoints of vertices and edges</u>

- Each vertex never gets white again after initialization.
- Each vertex gets into Q and gets out from Q at most once
- Each edge is checked at most once
  - when one endpoint vertex is taken from Q and its neighbors are checked along edges
- $\therefore O(|V|+|E|)$

```
BFS(V,E,s){
  for v∈V do
    toWhite(v);
  endfor
  toGray(s);
  Q={s};
  while( Q!={} ){
    u=pop(Q);
    for v∈{v∈V|(v,u)∈E}
      if isWhite(v) then
        toGray(v);
        push(Q,v);
      endif
    endfor
    toBlack(u);
  }}
```

Application of BFS: Shortest path problem on graph

Definition of "distance"

- Start vertex v has distance 0
- Except start vertex, each vertex u has distance d+1, where d is the distance of parent of u.

- On BFS, modify that each gray vertex receives its "distance" from black neighbor, then you get (shortest) distance from v to it.

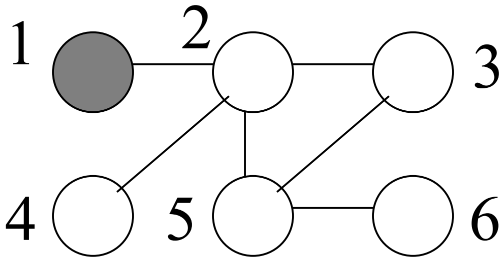
DFS (Depth-First Search)

- For a graph G=(V,E) and start point s∈V, it follows reachable vertices from s until it reaches a vertex that has no unvisited neighbor, and returns to the last vertex that has unvisited neighbors.

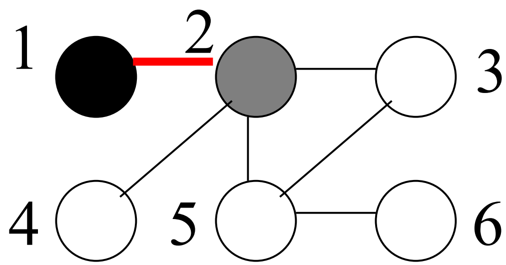
```
dfs(V, E, s) {
    visit(s)
    for  (s, w)∈E  do
        if  notVisited(w)  then
            dfs(V, E, w)
}
```

Program code is relatively simple, and vertices are put into a stack when dfs makes a recursive call.

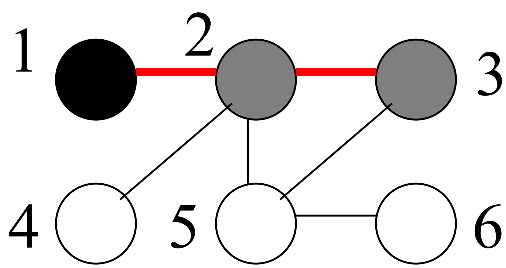
DFS: Example



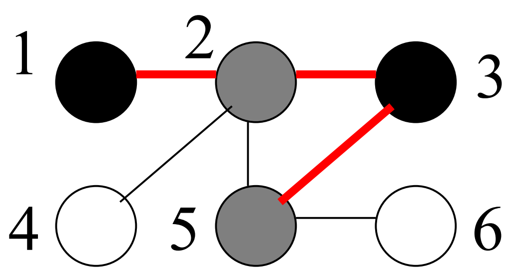
DFS(1)



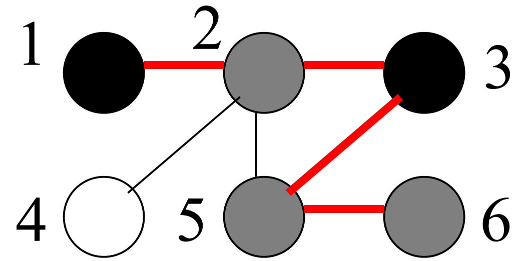
DFS(2)



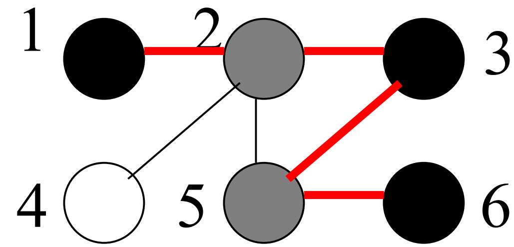
DFS(3)



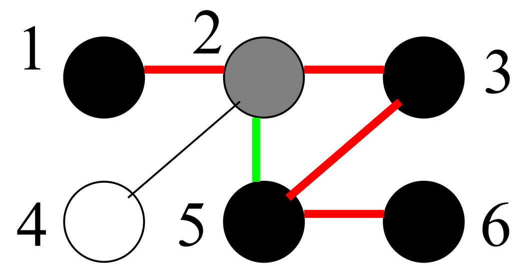
DFS(5)



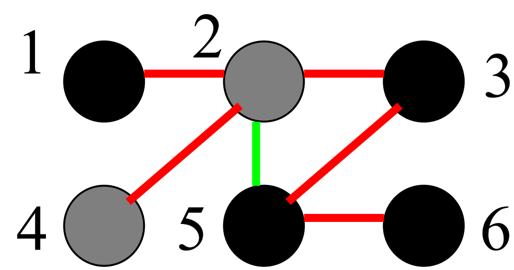
DFS(6)



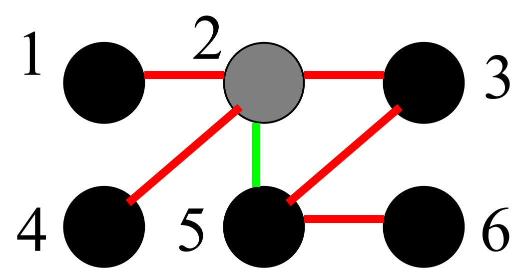
DFS(6)



DFS(5)
DFS(3)
DFS(2)



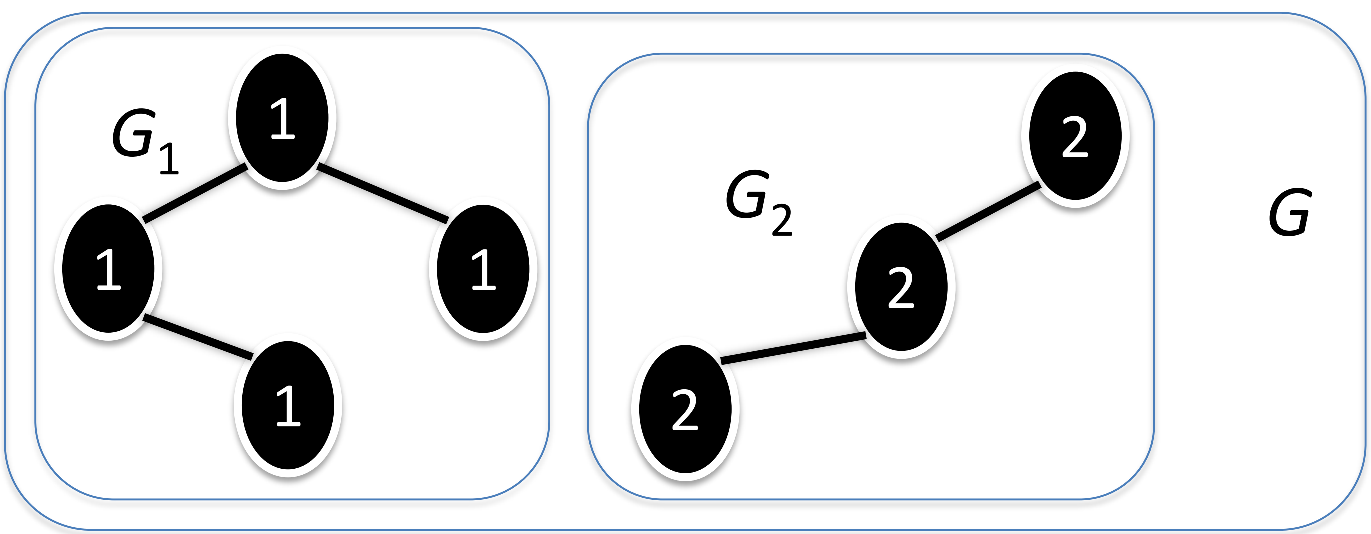
DFS(4)



DFS(2)

Application of DFS: Find connected components in a graph

- For a given (disconnected) graph $G = (V, E)$, divide it into connected graphs $G_1 = (V_1, E_1)$, …, $G_c = (V_c, E_c)$.
  - We will give a numbering array cn[] such that $\forall u,v \in V,\ u \in V_i \land v \in V_j \land i \neq j \Rightarrow cn[u] \neq cn[v]$



Application of DFS: Find connected components of a graph

```
cc(V,E,cn){ //cn[|V|]
  for v∈V do
      cn[v] = 0; /*initialize*/
  endfor
  k = 1;
  for v∈V do
    if cn[v]==0 then
      dfs(V,E,v,k,cn);
      k=k+1;
    endif
  endfor
}
```

```
dfs(V,E,v,k,cn){
  cn[v]=k;
  for u∈{u|(v,u)∈E} do
    if cn[u]==0 then
      dfs(V,E,u,k,cn);
    endif
  endfor
}
```

BFS v.s. DFS on a graph

- Two major (efficient) algorithms:
  - Breadth First Search:

    It corresponds to "Queue"
  - Depth First Search:

    It corresponds to "Stack"
  - Both algorithms are easy to implement to run in $O(|V|+|E|)$ time. (In a sense, this time complexity is optimal since you have to check all input data.)
  - Depending on applications, we choose better algorithm.