C Version

I111E Algorithms & Data Structures
3. Basic Programming

School of Information Science

Ryuhei Uehara & Giovanni Viglietta

uehara@jaist.ac.jp & johnny@jaist.ac.jp

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All materials are available at

http://www.jaist.ac.jp/~uehara/couse/2019/i111e

Main topic:

SEARCH PROBLEM

Search Problem

- Problem: S is a given set of data. For any given data x, determine efficiently if S contains x or not.

- Efficiency: Estimate the time complexity by $n = |S|$, the size of the set S
  - In this problem, "checking every data in S" is enough, and this gives us an upper bound O(n) in the worst case.
  - Can we do better?
  - How about *dictionary*?

Roughly, "the running time is proportional to n."

How to tackle the problem

- Consider data structure and how to store data
  - Data are in an array in any ordering
  - Data are in an array in increasing order
- Search algorithm: The way of searching
  - Sequential search
  - m-block method
  - Double m-block method
  - Binary search
- Analysis of efficiency
  - (Big-O notation)

We introduce these methods (m-block method, Double m-block method) to explain our naïve idea.

Data structure 1
Data are stored in arbitrary ordering

- Each element in the set *S* is stored in an array s from s[0] to s[*n*-1] in any arbitrary ordering.

s[ ]=									
	37	12	25	9	87	33	65	3	29

Sequential search

- Input: any natural number *x*
- Output:
  - If there is i such that s[i] == *x*, output i
  - Otherwise, output -1 (for simplicity)

```
for (i=0; i<n; ++i)
   if(x==s[i]) return i;
return -1;
```

In the worst case, we need *n* comparisons.
Thus, the running time is proportional to *n*.
→ O(*n*) time algorithm

Example: Real code of seq. search

```
public class i111_03_p7{
    public static void Main(){
        int[] data = new int[]{37,12,25,9,87,33,65,3,29};
        int len = data.Length;

        int target = 87;
        int result = find(target,len,data);
        if (result == -1) {
            System.Console.WriteLine(target+" not found");
        } else {
            System.Console.WriteLine(target+" is at index "+result);
        }
    }

    static int find(int x, int n, int[] s) {
        for (int i=0; i<n; i++) {
            System.Console.Write(i+" ");
            if (x==s[i]) return i;
        }
        return -1;
    }
}
```

Precise time complexity of sequential search

- At most 3n + 2 steps

Initialization of i takes 1 operation

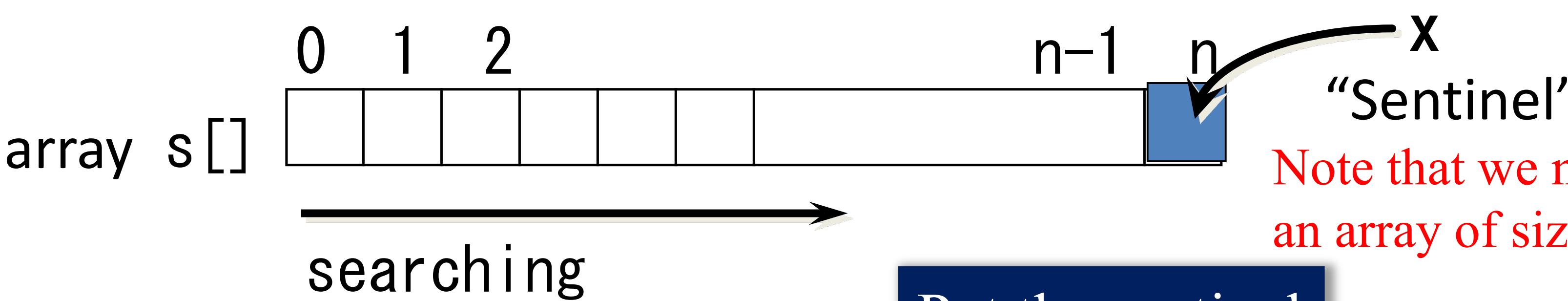
```
for (i=0; i<n; ++i)
    if(x==s[i]) return i;
return -1;
```

For the number of loops ≦ n,
  comparison × 2 (==, <)
  increment × 1 (++)

Return takes 1 operation

Programming tips 1: simplify by using “sentinel”

Before searching, push *x* itself at the end of the array;
Then you definitely have x==s[i] for some 0<=i<=n
So you do not need the check i<n any more.



Note that we need an array of size n+1

Put the sentinel

```
s[n] = x;
i = 0;
while(x != s[i])
 i = i+1;
if(i < n) return i;
else      return -1;
```

Simple loop!
→ 2 operations

At most 2n+4 (<3n+2) operations $=O(n)$

Analysis of the number of comparisons

```
s[n] = x;
i = 0;
while(x!=s[i])
 i = i+1;
if(i < n)
  return i;
else
  return -1;
```

Consider best/worst/average cases

- The best case: 1
  - when s[0] == x
- The worst case: n
  - when x is not in s[0]…s[n-1]
- The average case : $\sum_{i=1}^{n+1} \frac{i}{n} = \frac{n+2}{2}$
  - The expected value of # of comparisons
  - The i-th element is compared with probability 1/n
  - The number of comparisons when x is equal to the i-th element is i.

※average is close to *n* when we often have the case that *x* is not in data
※It depends on the situation that which case is important

What happens
if we use
“nice” data structure?

Data structure 2

Data in the array in increasing order

We don't consider how can we do now

- s[]=

3	9	12	25	29	33	37	65	87	*x*

- Q: Any improvement in sequential algorithm?

Idea

```
s[n]=x;
i = 0;
while(x!=s[i])
 i = i+1;
if(i < n) return i;
else      return -1;
```

We can stop when s[i] is greater than x
x!=s[i] → x>s[i]

Data structure 2

Data in the array in increasing order

We don’t consider how can we do now

- s[]=

3	9	12	25	29	33	37	65	87	*x*

- Q: Any improvement in sequential algorithm?

```
s[n]=x;
i = 0;
while(s[i]<x)
 i = i+1;
if(i < n) return i;
else      return -1;
```

It does not happen over *x*!

Idea

We can stop when s[i] is greater than x

x!=s[i] → x>s[i]

Data structure 2

Data in the array in increasing order

- s[]=

3	9	12	25	29	33	37	65	87	*x*

- Q: Any improvement in sequential algorithm?

```
s[n]=x;
i = 0;
while(s[i]<x)
 i = i+1;
if(i < n) return i;
else      return -1;
```

We can stop when s[i] is greater than x
x!=s[i] ➔ x>s[i]

It may stop even if i<n
i<n ➔ s[i]==x
E.g, if x=30, we have i<n (5<9)
but it should return (−1)

Look!

Data structure 2
Data in the array in increasing order

- s[]=

3	9	12	25	29	33	37	65	87

x

- Q: Any improvement in sequential algorithm?

```
s[n]=x;
i = 0;
while(s[i]<x)
 i = i+1;
if(s[i]==x) return i;
else        return -1;
```

Much intuitive condition!

We can stop when s[i] is greater than x
x!=s[i] → x>s[i]

It may stop even if i<n
i<n → s[i]==x

Data structure 2

Data in the array in increasing order

- s[]=

3	9	12	25	29	33	37	65	87

x

Look!

- Q: A … ential algorithm?

When x is not in s[],
it returns n
s[n]=x → s[n]=x+1

```
s[n]=x;
i = 0;
while(s[i]<x)
 i = i+1;
if(s[i]==x) return i;
else        return -1;
```

We can stop when s[i] is greater than x
x!=s[i] → x>s[i]

It may stop even if i<n
i<n → s[i]==x

Data structure 2
Data in the array in increasing order

- s[]=

3	9	12	25	29	33	37	65	87	*x*+1

- Q: A … equential algorithm?

When x is not in s[], it returns n
s[n]=x → s[n]=x+1

```
s[n]=x+1;
i = 0;
while(s[i]<x)
 i = i+1;
if(s[i]==x) return i;
else        return -1;
```

We can stop when s[i] is greater than x
x!=s[i] → x>s[i]

It may stop even if i<n
i<n → s[i]==x

Data structure 2
Data in the array in increasing order

- s[]=

3	9	12	25	29	33	37	65	87	*x*+1

  - Exit from loop when: s[i]≧x
  - Check after loop: s[i]==x
  - Sentinel: greater than x, e.g., x+1

```
s[n]=x+1;
i = 0;
while(s[i]<x)
 i = i+1;
if(s[i]==x) return i;
else        return -1;
```

Q. Improve of comparison?

A. Average is better.
But the same in
the worst case

Q : When the average is better?

Example: Real code of seq. search in increasing order

```
public class i111_03_p18{
    public static void Main(){
        int[] data = new int[]{3,9,12,25,29,33,37,65,87,-1};
        int len = data.Length-1;

        int target = 17;
        int result = find(target,len,data);
        if (result == -1) {
            System.Console.WriteLine(target+" not found");
        } else {
            System.Console.WriteLine(target+" is at index "+result);
        }
    }

    static int find(int x, int n, int[] s) {
        s[n] = x+1;
        int i=0;
        while (s[i]<x) {
            System.Console.Write(i+" ");
            i++;
        }
        if (x==s[i]) return i;
        return -1;
    }
}
```

Minor improvements of number of comparisons in sequential search

(Tips 1)

In the array, the minimum data is the first, and the maximum data is the last. Thus, depending on x and them, we can change the direction of search.

➔ We still need n-1 comparisons in the worst case

(Tips 2)

First, compare x with the medium data s[n/2]. If x is larger, search the right half, and search the left half otherwise.

➔At most n/2 comparisons. Much smaller.

➔It is still $O(n)$, but,,,

Drastic improvement from $O(n)$!!

Drastic Improvement from O(n)

Algorithm 2: m-block method

Idea of m-block method

(0) Divide the array into m blocks B_0, B_1, ... , B_{m-1}

(1) Check the biggest item in each block,
and find the block B_j that can contain x

(2) Perform sequential search in B_j

Algorithm 2: m-block method

Idea of m-block method

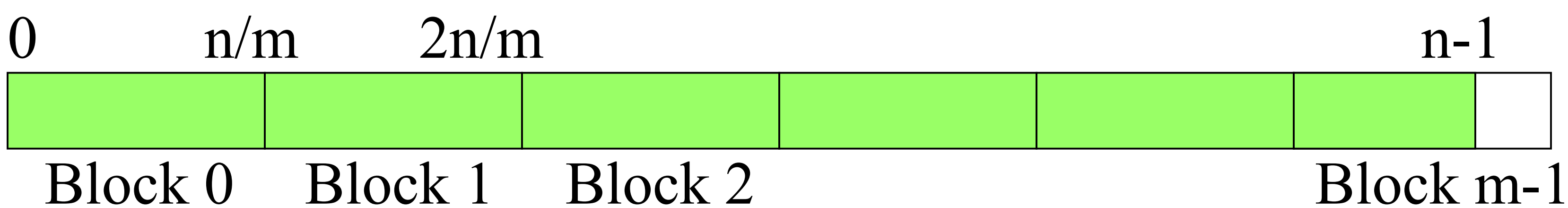
(0) Divide the array into m blocks $B_0, B_1, \ldots, B_{m-1}$

(1) Check the biggest item in each block, and find the block B_j that can contain x

(2) Perform sequential search in B_j

Simple implementation:

divide into the blocks of same size except the last one.



- Each block has length k, where $k = \lceil n/m \rceil$
- Block B_j has items from s[jk] to s[(j+1)k-1]: B_j = [jk, (j+1)k-1]

Algorithm 2: m-block method

Idea of m-block method

(0) Divide the array into m blocks $B_0, B_1, \ldots, B_{m-1}$

(1) Check the biggest item in each block,
 and find the block B_j that can contain x

(2) Perform sequential search in B_j

```
j=0;
while(j<=m-2)
   if x>=s[(j+1)*k-1] then exit from loop
   else j=j+1;
```

j=0,…,m-2, m-1 is "leftover"

The maximum index of B_j

If the program exits from the loop, the variable j indicates the index of the block, and j indicates the last one otherwise.

Algorithm 2: m-block method

Idea of m-block method

(0) Divide the array into m blocks $B_0, B_1, \dots, B_{m-1}$

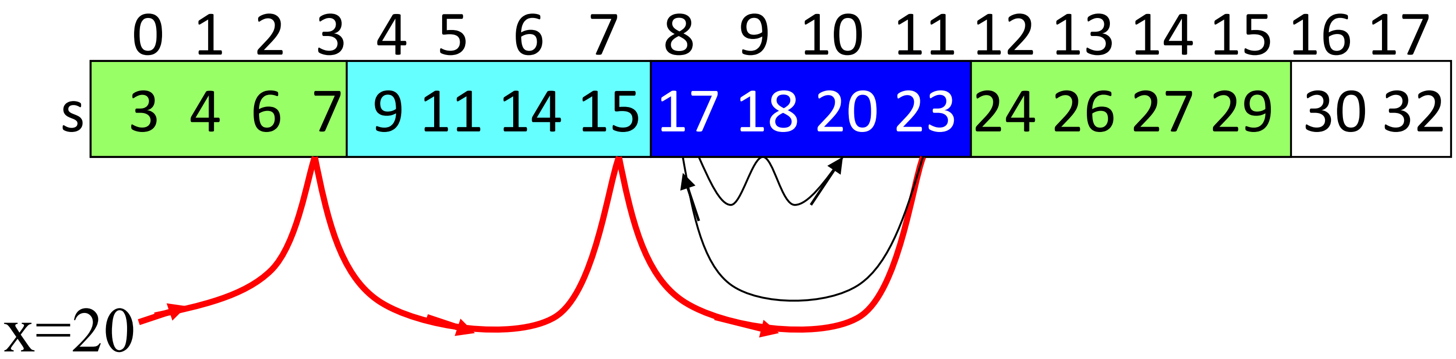
(1) Check the biggest item in each block,
and find the block B_j that can contain x

(2) Perform sequential search in B_j

```
i=j*k; t = min{ (j+1)*k-1, n-1 };
while( i < t )
  if x≧s[i] then exit from the loop;
  else i=i+1;  //next item in the block
if x == s[i] then return i and halt;
else  return -1 and halt.
```

Note that we cannot use sentinel since we have no extra space between block

Example and time complexity



- # of comparisons $\leqq$ # of blocks $+$ length of block = $m + n/m$
- What the value of m that minimize $m + n/m$?
  - Let $f(m) = m + n/m$, and take the differential for m
  - $f'(m) = 1 - n/m^2 = 0 \rightarrow m = \sqrt{n}$
  - When $m = \sqrt{n}$, # of comparisons $\leqq \sqrt{n} + n/\sqrt{n} = 2\sqrt{n}$
- Time complexity: $O(\sqrt{n})$

5 min. ex.
Assume n=100.
Find "average" and "worst" cases for m=10, m=2, and m=50

For example, when n=1000000,
Linear search takes n/2=<u>500000</u> comparisons, but
Block search takes $\sqrt{1000000}$=<u>1000</u> comparisons!!

Example: Real code of m block method

```
public class i111_03_p27{
    public static void Main(){
         int[] data = new int[]{3,9,12,25,29,33,37,65,87};
         … the same as p7 … }

    static int find(int x, int n, int[] s) {
        int m=3;
        int k=(n-1)/m +1;

        int j=0;
        while (j<=m-2) {
            System.Console.Write(((j+1)*k-1)+" ");
            if (x<=s[(j+1)*k-1]) break;
            j++;
        }

        int i=j*k;
        int t=System.Math.Min((j+1)*k-1, n-1);
        while(i<t) {
            System.Console.Write(i+" ");
            if (x<=s[i]) break;
            i++;
        }
        if (x==s[i]) return i;
        return -1;
    }
}
```

【Pretty maniac】

Discussion of m block method

- Lengths of blocks should be the same?

(Observation)

# of comparisons ＝ # of searched blocks

＋ # of comparisons in the block

When you find former block, you can use more time in the block

➔It is better to decrease the length of blocks

・For example, we set $|B_{i+1}|=|B_i|-1$

・Make “index”+”length of a block” constant

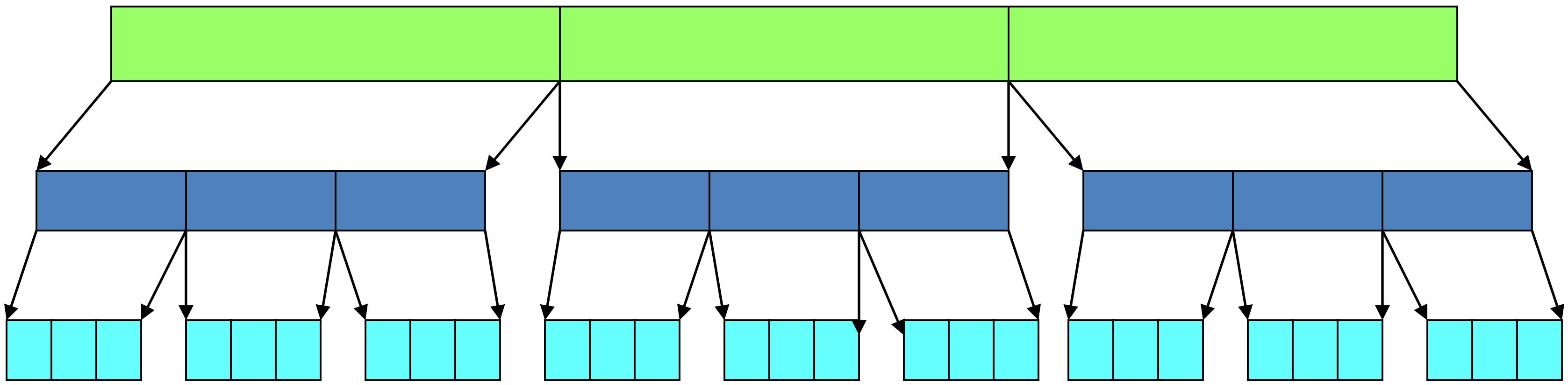
In reality, this kind of method of decreasing “unevenness” is preferred.

Can we do better than $O(\sqrt{n})$?

Algorithm 3: Double m-block method

In the m-block method, we use sequential search in each block.

⇒ We can use m-block method again in the block!!



Idea of double m-block method

For example, if the number of data is 27,

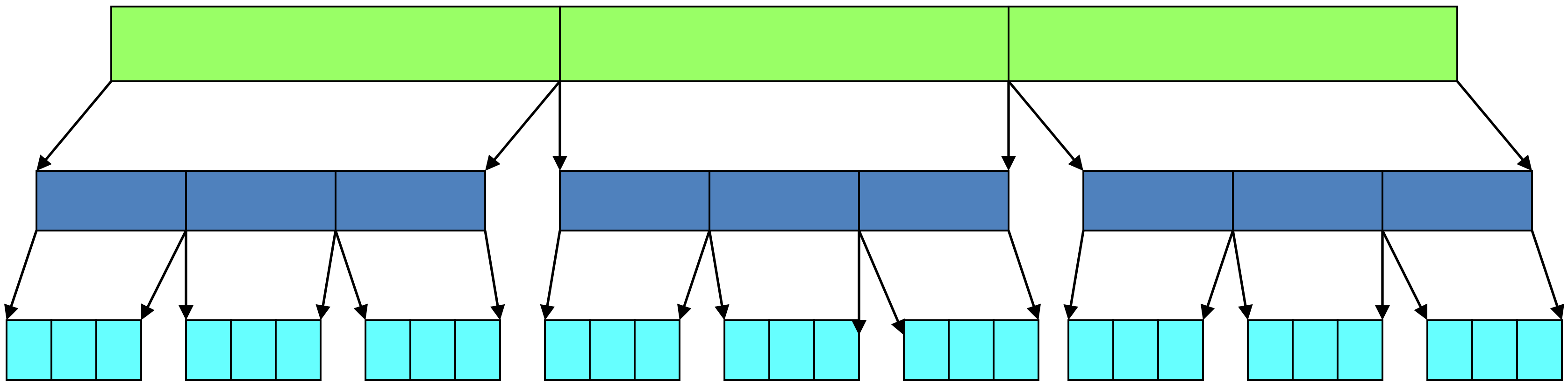
- Linear search requires 27 in the worst case
- 3-block method requires at most 3+9
- Double 3-block method needs at most 3+3+3

Algorithm 3: Double m-block method

In the m-block method, we use sequential search in each block.

→ We can use m-block method again in the block!!

Recursive call: basic and **strong** idea

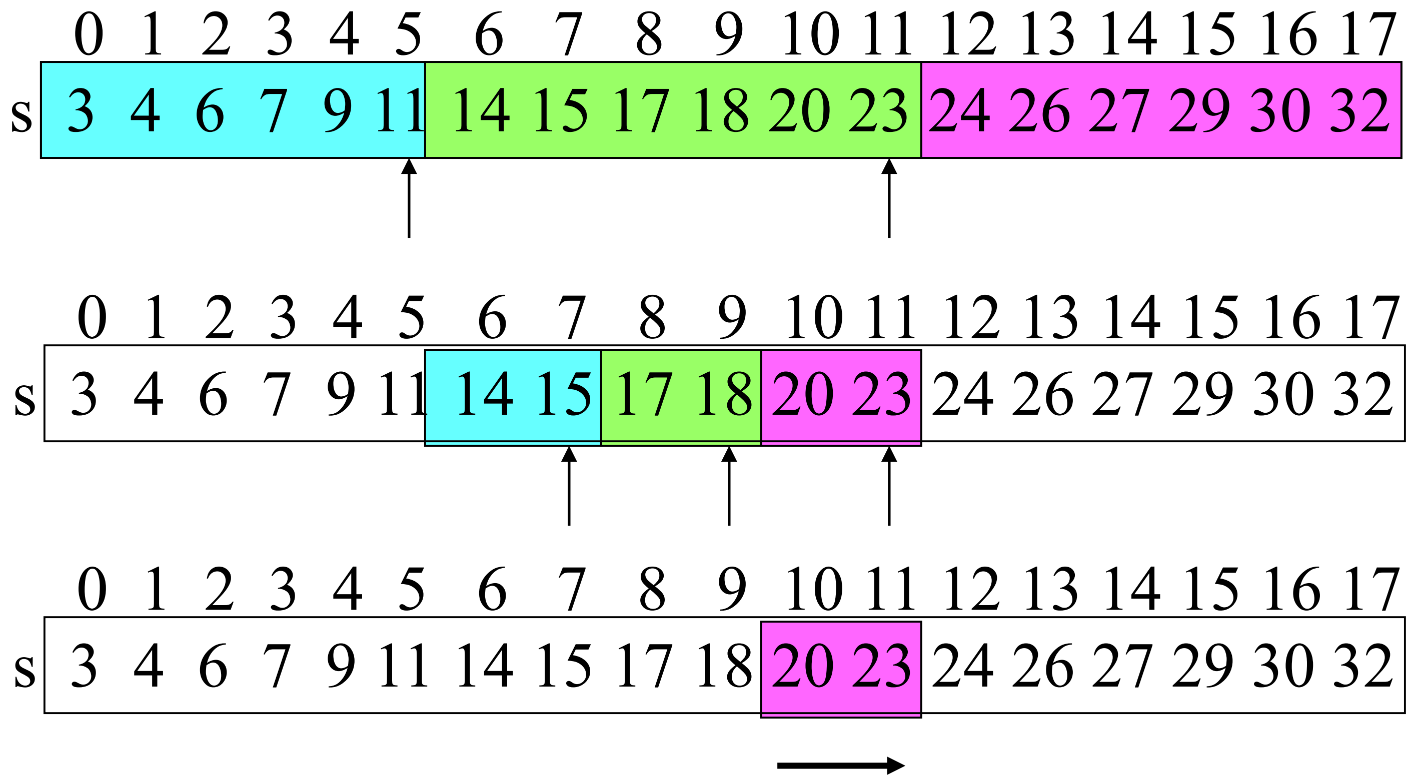


Idea of double m-block method

Why we stop only twice? We can more!!

Divide search area into m blocks, and repeat the same process for the block that contains x, and repeat again and again up to the block has length at most some constant N

Example:
find 20 (x=20) for block size 3



【I don’t ask you to compute it by yourself…】

Analysis of time complexity

- Length of search space

$$n \to \left\lceil \frac{n}{m} \right\rceil \to \left\lceil \frac{\left\lceil \frac{n}{m} \right\rceil}{m} \right\rceil \to \left\lceil \frac{\left\lceil \frac{\left\lceil \frac{n}{m} \right\rceil}{m} \right\rceil}{m} \right\rceil \to \cdots$$

- Let n_i be the length after the i-th call

$$n_1 = \left\lceil \frac{n}{m} \right\rceil \leq \frac{n}{m} + 1$$

$$n_2 = \left\lceil \frac{n_1}{m} \right\rceil \leq \frac{n}{m^2} + \frac{1}{m} + 1$$

$$\cdots$$

$$n_i \leq \frac{n}{m^i} + \sum_{j=0}^{i-1} \frac{1}{m^j} \leq \frac{n}{m^i} + 2$$

【I don’t ask you to compute it by yourself…】

Analysis of time complexity

- The length n_i after the i-th recursive call:

  $n_i \leqq n/m^i + 2$

- How many recursive calls made?

$$\mathrm{n_i} \leq \mathrm{Lmin} \Longleftarrow \mathrm{Lmin} \geq \frac{\mathrm{n}}{\mathrm{m^i}} + 2 \Longleftrightarrow \mathrm{i} \geq \log_{\mathrm{m}} \frac{\mathrm{n}}{\mathrm{Lmin} - 2}$$

- Each recursive call make at most m-1 comparisons, so the total number of comparisons is $\leq (\mathrm{m} - 1) \log_{\mathrm{m}} \frac{\mathrm{n}}{\mathrm{Lmin} - 2} + \mathrm{Lmin}$

- The time complexity is O(log *n*)

【I don't ask you to compute it by yourself…】

Analysis of time complexity: The best value of m

- $$T(n, m) = (m-1)\log_m \frac{n}{Lmin-2} + Lmin$$
  $$= \frac{m-1}{\log_2 m}\log_2 \frac{n}{Lmin-2} + Lmin$$
- To make T(n,m) the minimum, smaller m is better because m-1 grows faster than $\log_2 m$ (which will be checked in the big-O notation).
- Therefore, m=2 is the optimal

We will have "binary search"

[Summary]

- For unorganized data, we have to use O(n) time.
- If data are sorted in increasing order,
  - We can exit from the loop when we find the position of x
  - Improved to $O(\sqrt{n})$ with m-block method with $m=\sqrt{n}$
  - Improved to O(log n) with doubly m-block method with m=2
- Honestly, in recent programming environment, you do not need to make such a search by yourself.
- Usually, we use a function indexOf(). However, it is very important that you should know that
  - "indexOf is heavy" for unorganized data
  - "indexOf is light" for SortedList