Introduction to Algorithms and Data Structures

Lecture 11: Sorting (2)
Heap sort and Merge sort

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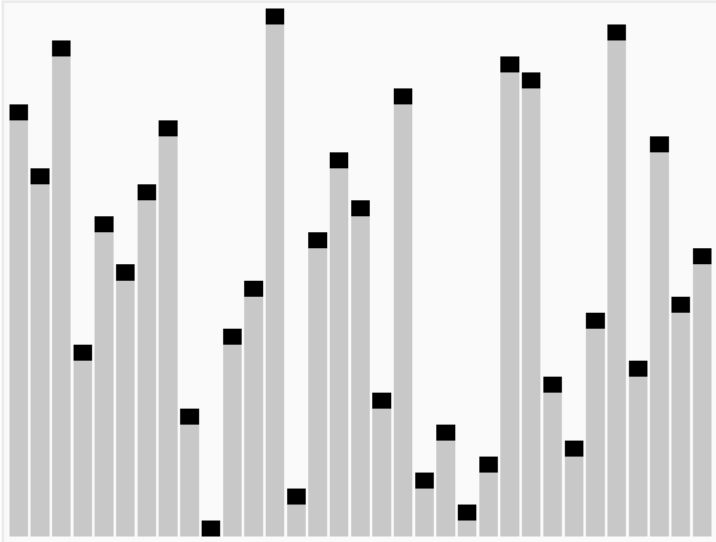


HEAP SORT

Heap sort

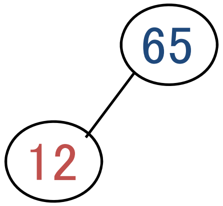
- Data structure heap
  - Insertion of data: $\Theta(\log n)$ time
  - Take the maximum element: $\Theta(\log n)$ time
- How to sort by heap
  - Step 1: Put *n* elements into heap
  - Step 2: Repeat to take the maximum element from heap, and copy it to the rightmost element
- Computational Complexity:
  - Both of steps 1 and 2 take $\Theta(n \log n)$ time.

Example of heap sort @Step 1

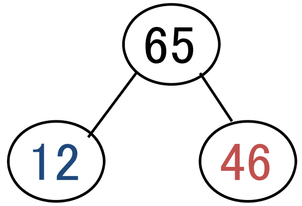
Data = 65 12 46 97 56 33 75 53 21

(1)add 65

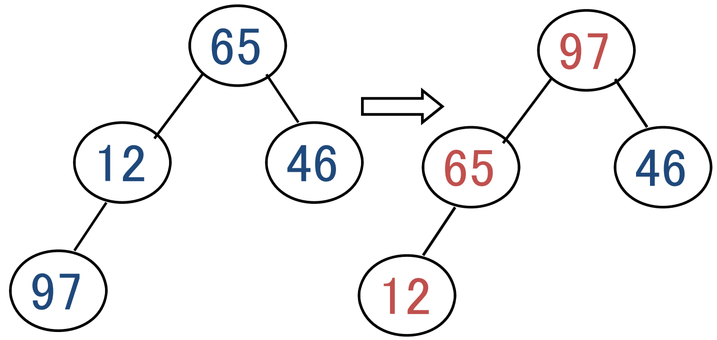
(2)add 12



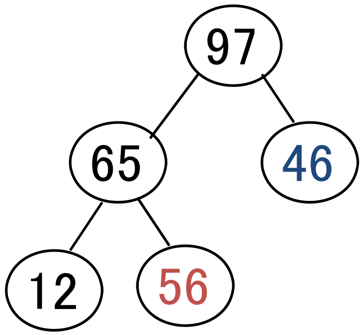
(3)add 46



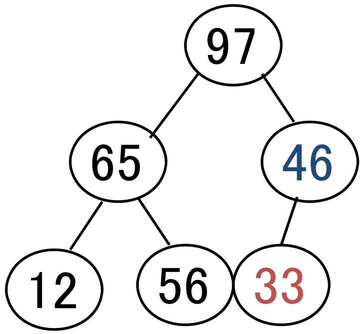
(4)add 97



(5)add 56



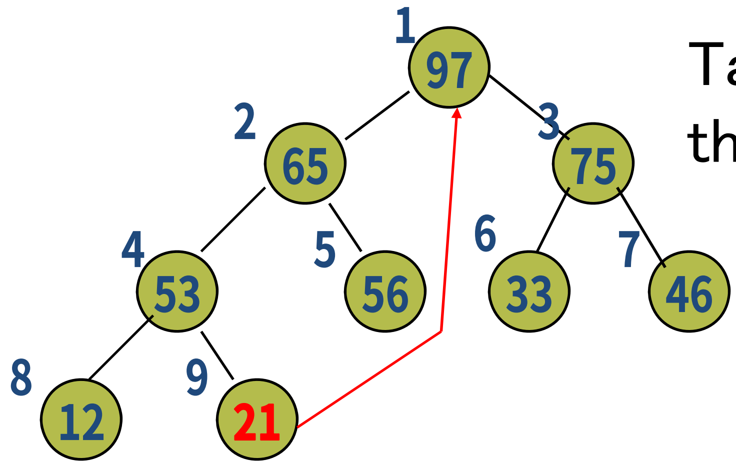
(6)add 33



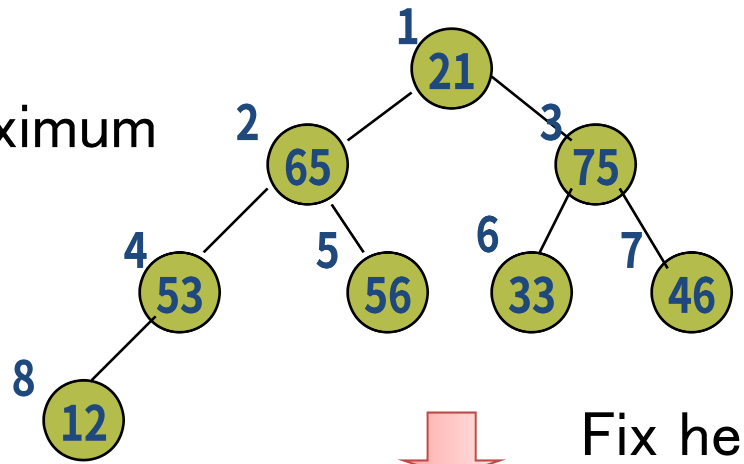
… in the same way, we can add data to heap one by one:

1	2	3	4	5	6	7	8	9
97	65	75	53	56	33	46	12	21

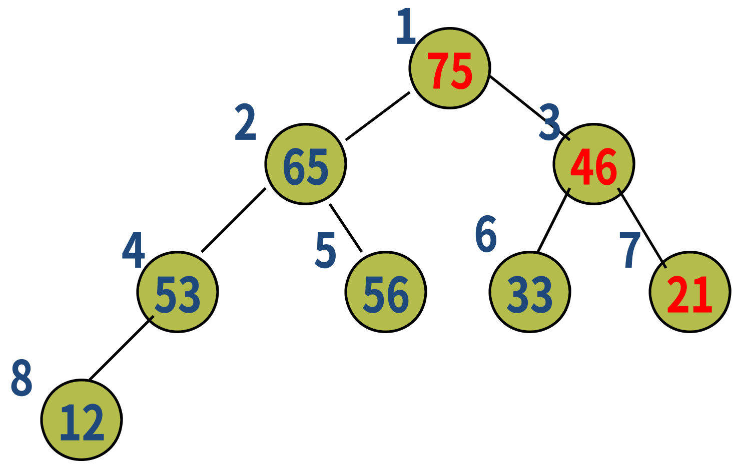
Example of heap sort @Step 2



Take
the maximum



Fix heap



75	65	46	53	56	33	21	12	

Copy to the
rightmost
element

75	65	46	53	56	33	21	12	97

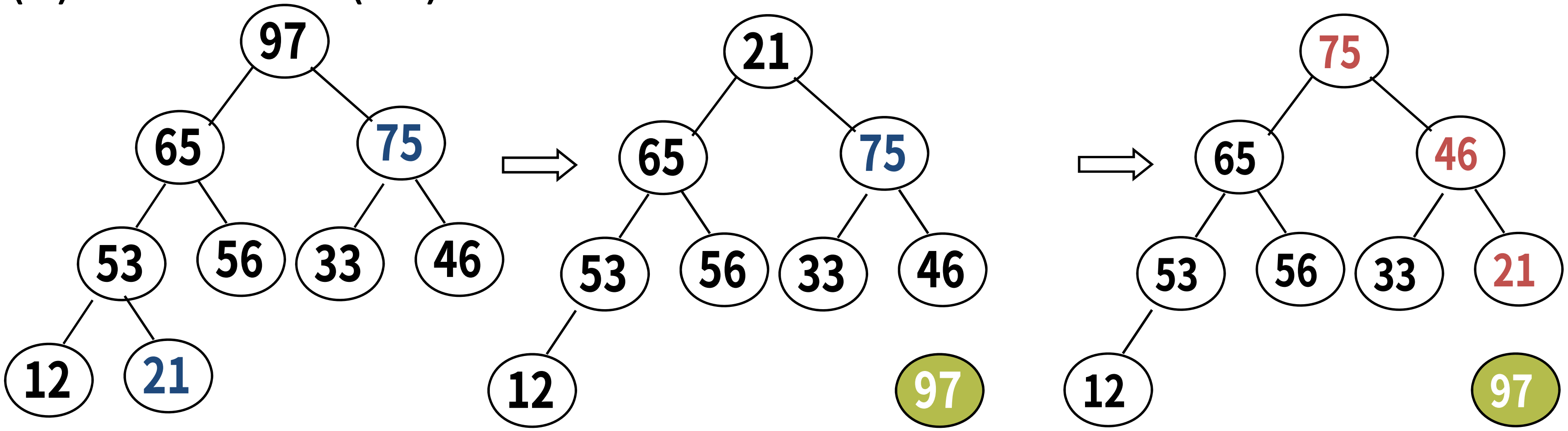
Example of heap sort @Step 2

array =

97	65	75	53	56	33	46	12	21

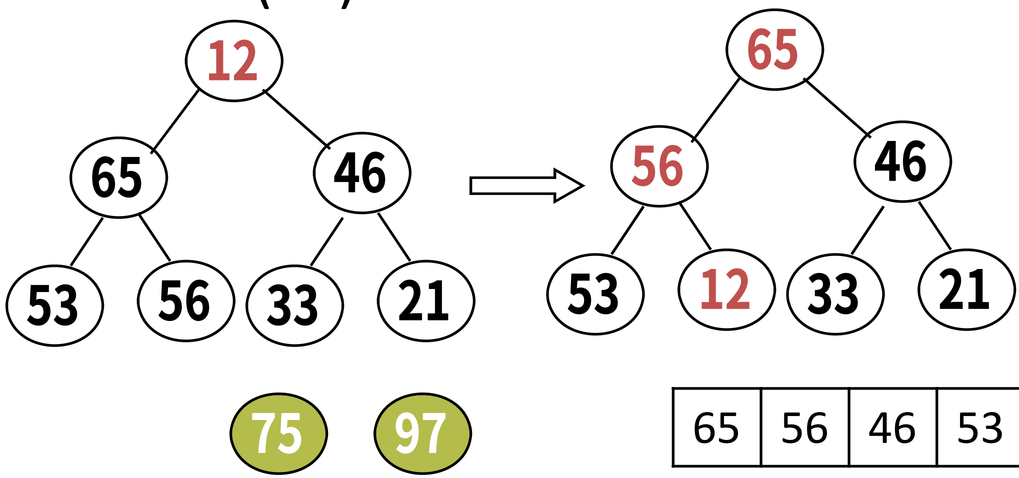
:

(1)delete max (97)



75	65	46	53	56	33	21	12	97

(2) delete max (75)

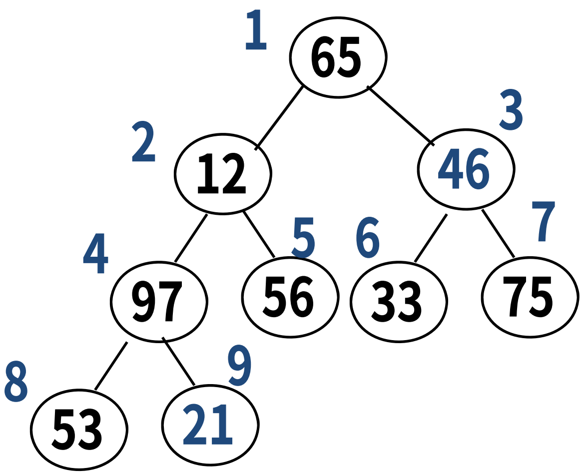


65	56	46	53	12	33	21	75	97

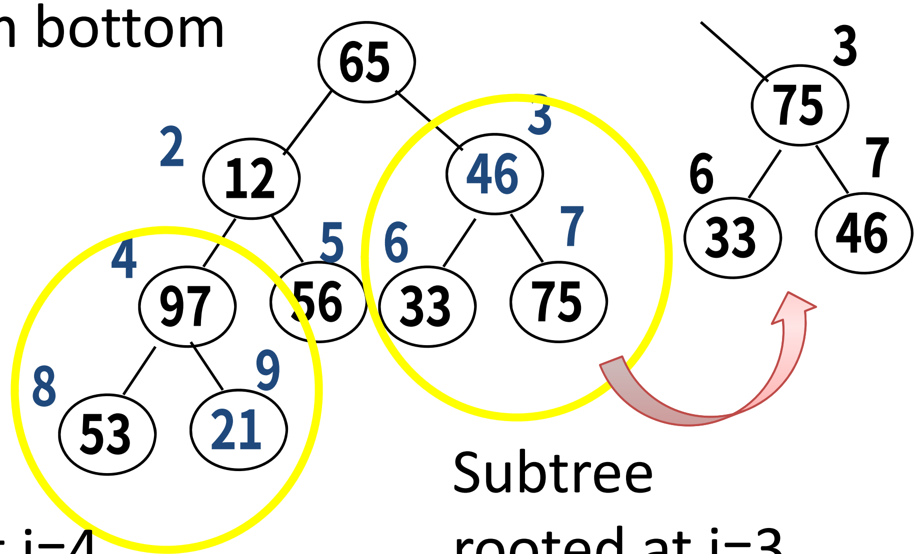
(Bit) improvement of heap sort

- We can make step 1 to run in $\Theta(n)$ time
  - Add all items into the array first
  - From bottom to top, exchange the parent/child

(1) Store data



(2) Exchange data in each parent/child from bottom



Subtree rooted at i=4

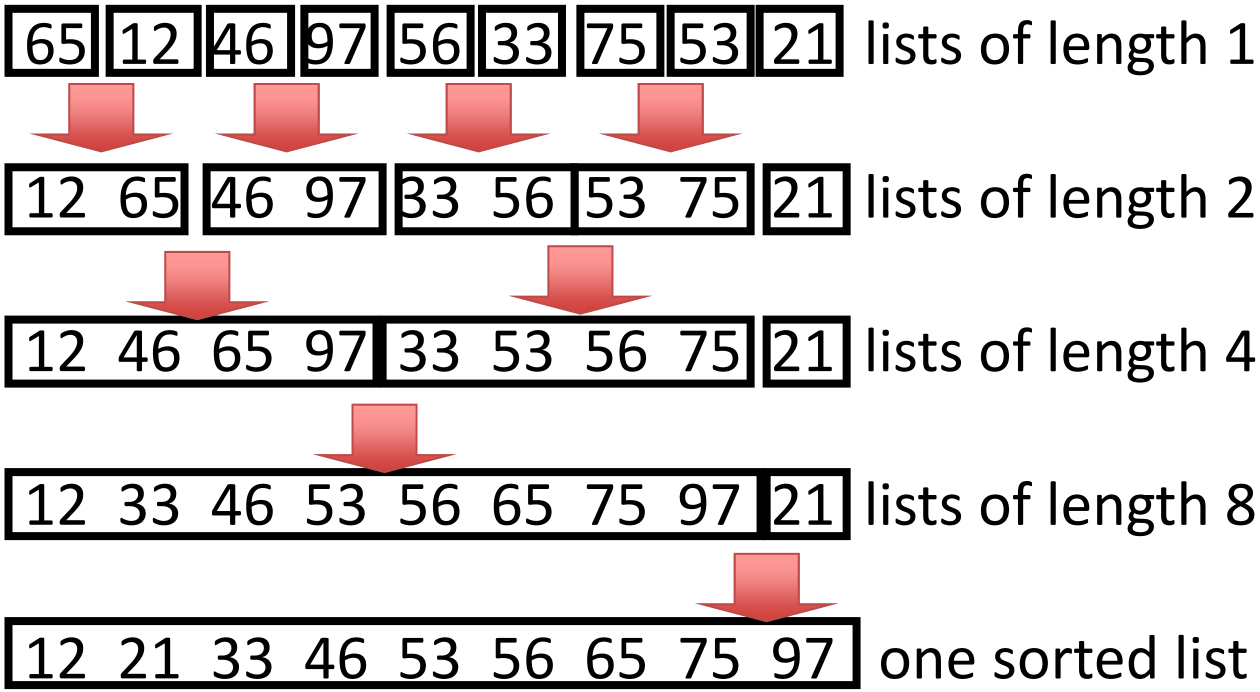
Subtree rooted at i=3

John von Neumann
1903−1957

MERGE SORT

Merge sort

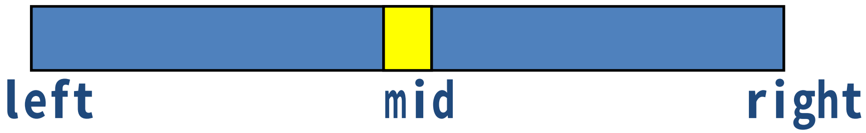
- It repeats to <u>merge</u> two sorted lists into one (sorted) list



- First, it repeats to divide until all lists have length 1, and next, it merges each two of them.

Implementation of merge sort: Typical recursive calls

- The interval that will be sorted: [left, right]
- Find center mid = (left + right)/2



- [left,right]➔[left,mid], [mid+1,right]
- Perform merge sort for each of them, and merge these sorted lists into one sorted list.

Outline of merge sort

```
MergeSort(int left, int right){
   int mid;
   if(interval [left,right] is short)
     (sort by any other simple sort algorithm);
   else{
     mid = (left+right)/2;
     MergeSort(left, mid);
     MergeSort(mid+1, right);
     Merge [left, mid] and [mid+1, right];
   }
}
```

We can merge two lists of length p and q in $O(p + q)$ time.

Merge sort: the merge process

To merge [left, mid] and [mid+1, right]:

```
i=left; j=mid+1; k=left;
while(i<=mid && j<=right)
  if(a[i] <= a[j]) {
    b[k]=a[i]; k++; i++;
  } else {
    b[k]=a[j]; k++; j++;
  }
while(j<=right){ b[k]=a[j]; k++; j++; }
while(i<=mid){ b[k]=a[i]; k++; i++; }
for(i=left; i<=right; i++) a[i]=b[i];
```

$O(p+q)$ time (for the two-way `while` loop)

Temporarily, it stores items in a[] to b[] to merge.

Write back b[] to a[]

Merge sort: Time complexity

- $T(n)$: Time for merge sort on n data
  - $T(n) = 2T(n/2)$ + "time to merge"
    $= 2T(n/2) + cn + d$ (c, d: some positive constant)
- To simplify, letting $n = 2^k$ for integer k,

$$
\begin{aligned}
T(2^k) &= 2T(2^{k-1}) + c2^k + d \\
&= 2(2T(2^{k-2}) + c2^{k-1} + d) + c2^k + d \\
&= 2^2T(2^{k-2}) + 2c2^k + (1+2)d \\
&= 2^2(2T(2^{k-3}) + c2^{k-2} + d) + 2c2^k + (1+2)d \\
&= 2^3T(2^{k-3}) + 3c2^k + (1+2+4)d \\
&\vdots \\
&= 2^iT(2^{k-i}) + ic2^k + (1+2+\ldots 2^{i-1})d \\
&= 2^kT(2^0) + kc2^k + (1+2+\ldots 2^{k-1})d \\
&= bn + cn\log n + (n-1)d \in O(n\log n)
\end{aligned}
$$

Merge sort: Space complexity

- It is easy to implement by using two arrays a[] and b[].
  - Thus space complexity is $\Theta(n)$, or we need *n* extra array for b[].
  - It seems to be difficult to remove this “extra” space.
  - On the other hand, we can omit “Write back b[] to a[]” (in the 2 previous slides) when we use a[] and b[] alternately.

Maybe this “extra” space is the reason why merge sort is not used so often…

Monotone sequence merge sort

- Bit improved merge sort from the practical viewpoint.
- It first divides input into monotone sequences and merge them. (Original merge sort does not check the input)

Example: For 65, 12, 46, 97, 56, 33, 75, 53, 21;

[65 12] [46 97] [56 33] [75 53 21] Divide into monotone sequences

[12 46 65 97] [21 33 53 56 75] Merge neighbors

[12 21 33 46 53 56 65 75 97] Sorted!

Monotone sequence merge sort: Time complexity

- We can merge in O(p+q) time to merge two sequences of length p and q
- After merging, the number of sequences becomes in half.
  - When the number of monotone sequences is h, the number of recursion is $\log_2 h$ times.
- One recursion takes O(n) time

  → O(n log h) time in total.

- When data is already sorted: h = 1 → O(n) time
- The maximum number of monotone sequences is n/2

  → O(n log n) time in total.