# 実践的アルゴリズム理論
# Theory of Advanced Algorithms

## 計算折り紙(2)

### 担当:上原隆平

今後の予定:

28日(水):講義時間に最後の講義

- 講義アンケート(端末持参のこと)

28日(水):チュートリアルアワーに期末試験

- 試験範囲:後半重視+ちょっと前半
- 筆記用具, ノート, スライドのコピーはOK.

# Theory of Advanced Algorithms
# 実践的アルゴリズム理論

## Computational Origami (2)

**Ryuhei Uehara**

Schedule：

28 (Wed)：Last lecture

- Questionnaire (bring your note PC)

28 (Wed): Tutorial Hour: Final Examination

- Area: Mainly latter half + a bit from formar
- Pens & Pencils + Notes + Copies of Slides

# Today's Topic

1. **Folding 2 or more boxes from one polyomino**
   - **Relationship between polygon and convex polyhedron folded from it**
     - This problem is related to both of
       - Computational geometry
       - Graph theory and graph algorithms
     - We need "mathematical property", "nice algorithms", and "computer power"!

2. **Folding complexity of 1D origami**
   - **Fold 1 dimensional paper strip into unit length**
     - This problem is related to both of
       - Computational Complexity of algorithms
       - Enumeration and/or counting

# 1. Common developments of boxes

– Common developments that can fold to 2 different boxes.

– Common developments that can fold to 3 different boxes…

… and open problems

June, 2018



My result is used in main trick in a mystery (?) novel!

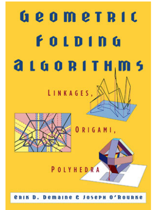
# 1. Common developments of boxes

References:


- Koichi Mizunashi, Takashi Horiyama, and Ryuhei Uehara: Efficient Algorithm for Box Folding, WALCOM 2019, March, 2019.
- Dawei Xu, Takashi Horiyama, Toshihiro Shirakawa, Ryuhei Uehara: Common Developments of Three Incongruent Boxes of Area 30, *COMPUTATIONAL GEOMETRY: Theory and Applications*, Vol. 64, pp. 1-17, August 2017.
- Toshihiro Shirakawa and Ryuhei Uehara: Common Developments of Three Incongruent Orthogonal Boxes, *International Journal of Computational Geometry and Applications*, Vol. 23, No. 1, pp. 65-71, 2013.
- Zachary Abel, Erik Demaine, Martin Demaine, Hiroaki Matsui, Guenter Rote and Ryuhei Uehara: Common Developments of Several Different Orthogonal Boxes, *Canadian Conference on Computational Geometry* (CCCG' 11), pp. 77-82, 2011/8/10-12, Toronto, Canada.
- Jun Mitani and Ryuhei Uehara: Polygons Folding to Plural Incongruent Orthogonal Boxes, *Canadian Conference on Computational Geometry* (CCCG 2008), pp. 39-42, 2008/8/13.

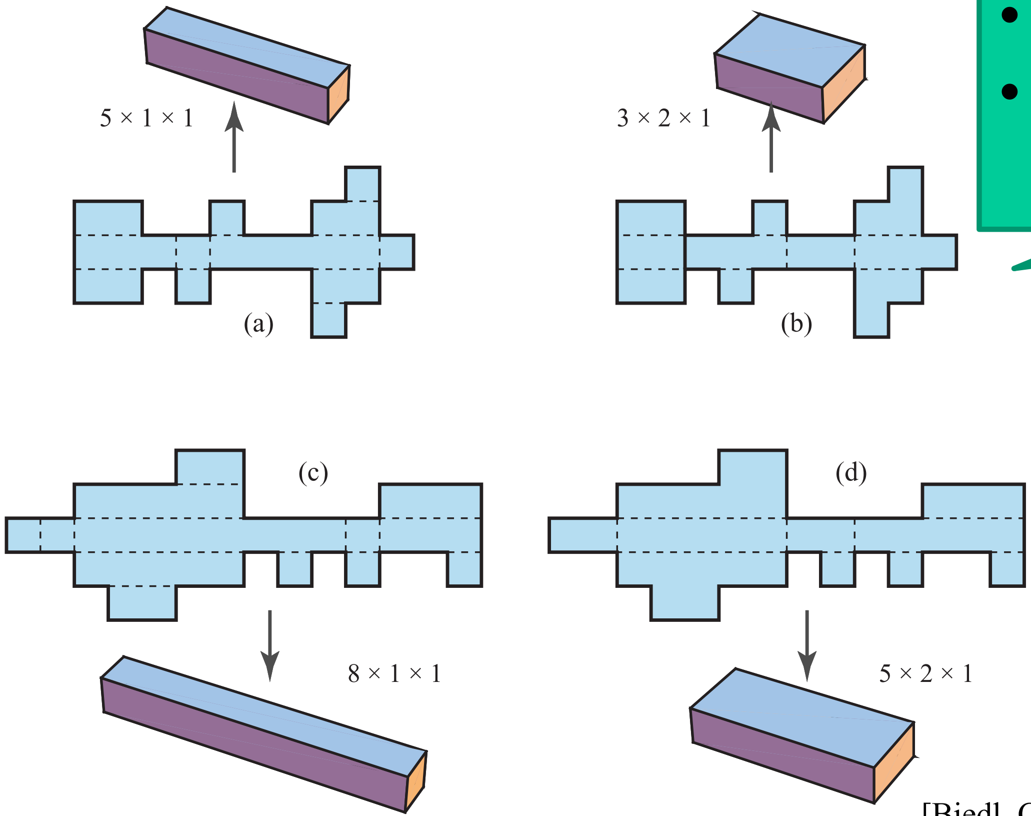

…and some developments:

http://www.jaist.ac.jp/~uehara/etc/origami/nets/index-e.html

# When I was translating  …



There are two polygons that can fold to
two different boxes;





- Are they “exceptional?”
- Polygons that fold to 3 or more boxes?

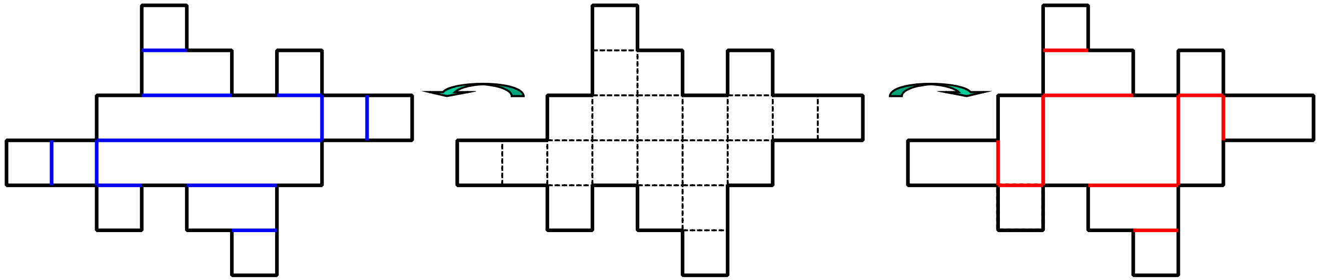
Biedl : I guess you cannot fold 3 boxes by one polygon…

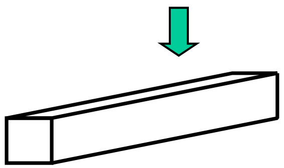
[Biedl, Chan, Demaine, Demaine, Lubiw, Munro, Shallit, 1999]

# Before computation…

Example
$1\times 1+1\times 5+1\times 5=1\times 2+2\times 3+1\times 3=11$ (Area: 22)

When a polygon can fold to 2 different boxes,

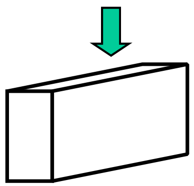




$1\times 1\times 5= a \times b \times c$

- We cut/fold along **unit squares** to simplify
- Surface area: $2(ab+bc+ca)$
- Necessary condition:

$ab+bc+ca=a'b'+b'c'+c'a'$



$1\times 2\times 3= a' \times b' \times c'$

*Good* areas have many 3-tuples

# Precomputation:
# Surface areas and possible size of boxes

If you want to find common developments of three boxes,

If you want to find common developments of four boxes,

| Area | 3-tuples | Area | 3-tuples |
|---|---|---|---|
| **22** | (1,1,5),(1,2,3) | 46 | (1,1,11),(1,2,7),(1,3,5) |
| 30 | (1,1,7),(1,3,3) | 70 | (1,1,17),(1,2,11),(1,3,8),(1,5,5) |
| **34** | (1,1,8),(1,2,5) | 94 | (1,1,23),(1,2,15),(1,3,11),(1,5,7),(3,4,5) |
| 38 | (1,1,9),(1,3,4) | 118 | (1,1,29),(1,2,19),(1,3,14),(1,4,11),(1,5,9),(2,5,7) |

Known results

# Polygons that fold to 2 boxes

In [Uehara, Mitani 2008], I ran a randomized algorithm that unfolds many target boxes of several sizes (infinitely :-)

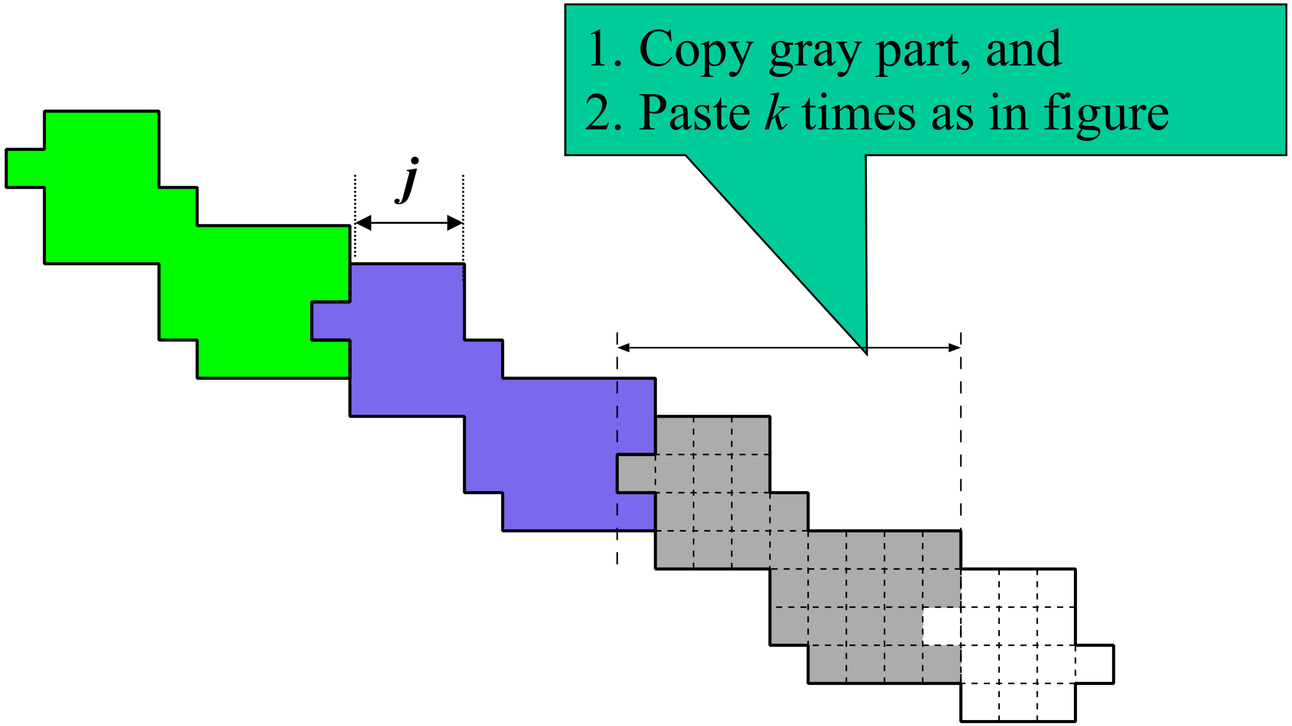
- That fold to 2 boxes;
  1. There are pretty many (∼9000)
     (by Supercomputer SGI Altix 4700)
  2. Theoretically,
     there are infinitely many!


- To 3 boxes…?

# Common developments of 2 boxes

[Theorem] There are infinitely many common developments of 2 boxes.

**[Proof]**



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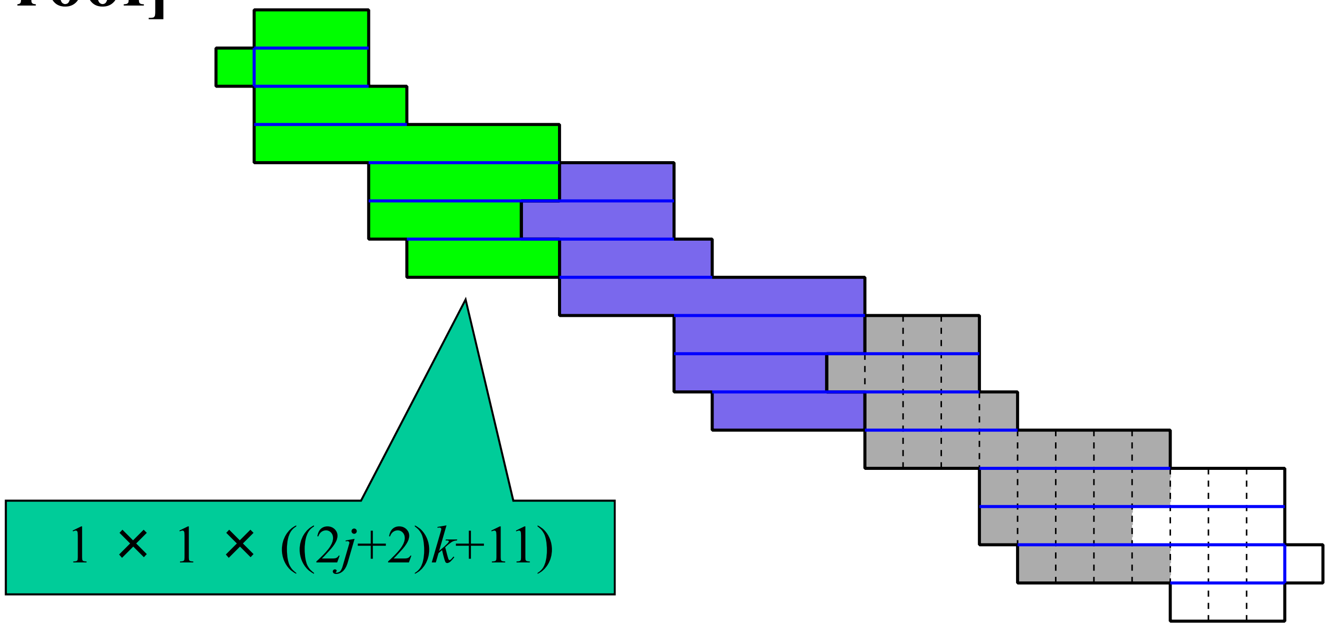

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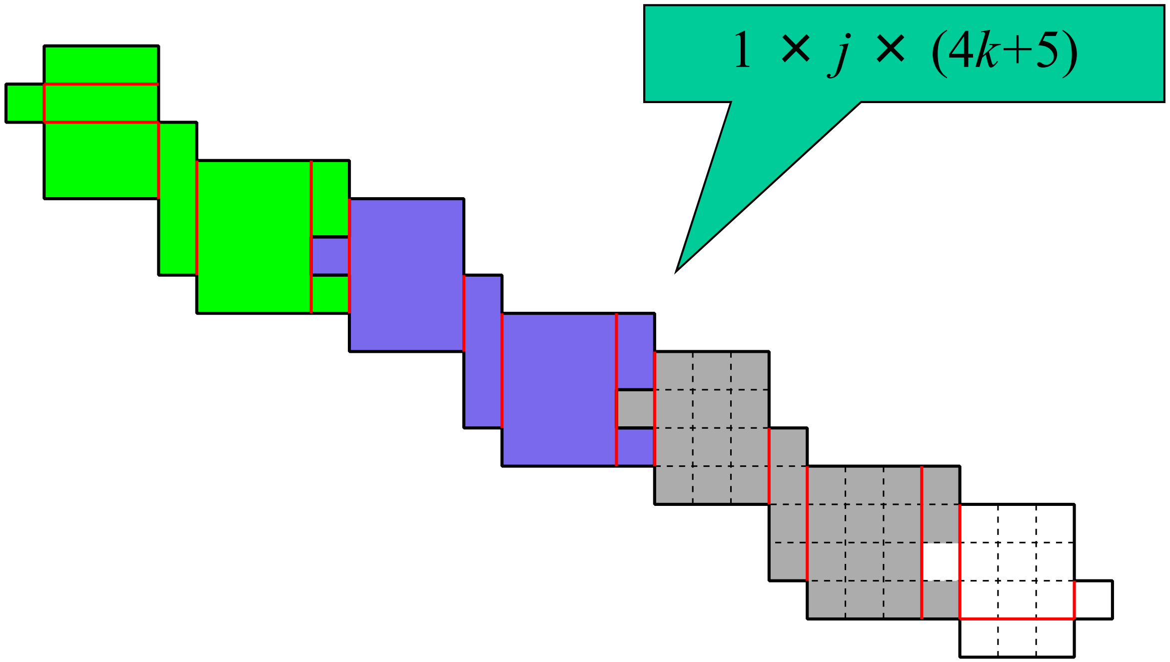
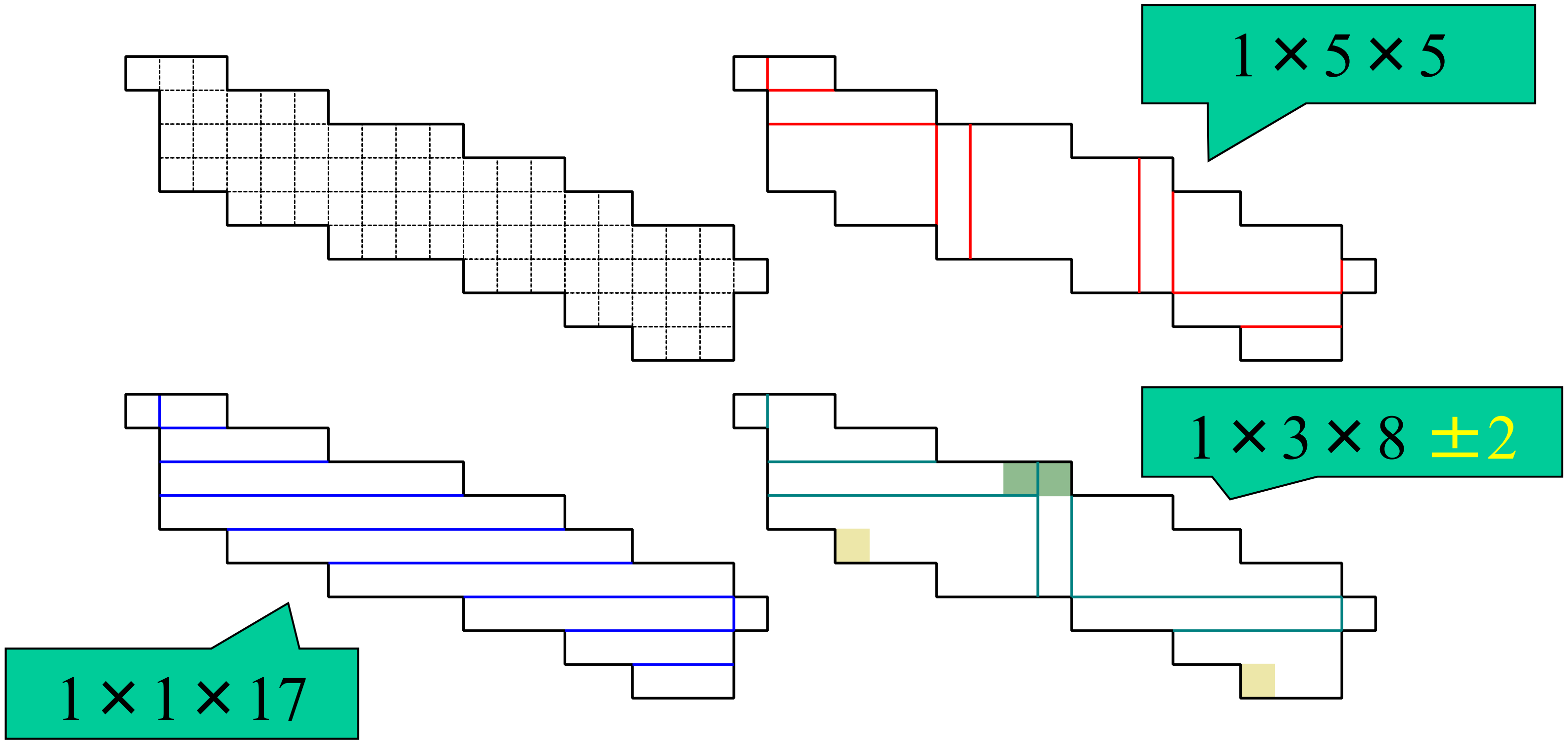


# Common development of 3 boxes?

Is there a common development of 3 boxes?

- Pretty close solution among 2 box solutions of area 46:



# Challenge to common development of 3 boxes

In [Abel, Demaine, Demaine, Matsui, Rote, Uehara 2011]

- The number of common developments of area 22 that fold into two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$ is 2263 in total.

  Program in 2011: It ran around 10 hours on a desktop PC.

- Among these 2263 common developments, there is only one pear development…

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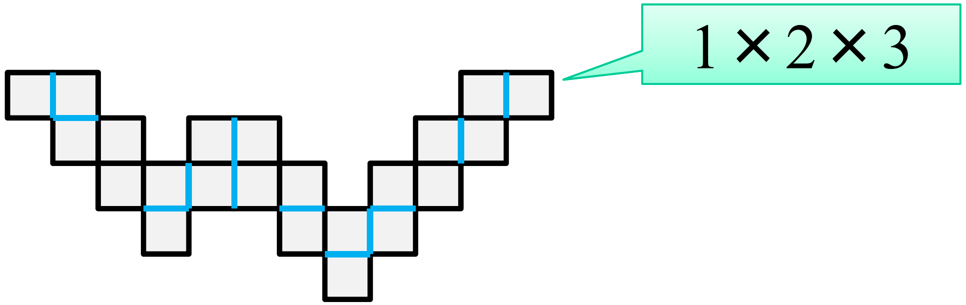

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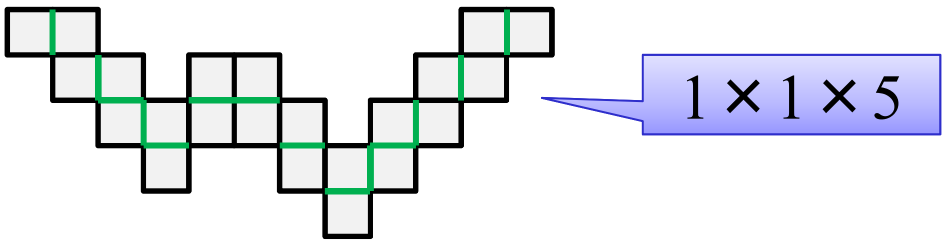

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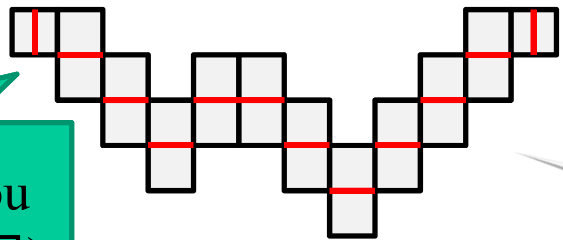
Is it cheating using "box" of volume 0?



Each column has 2 squares, so we can fold it vertically

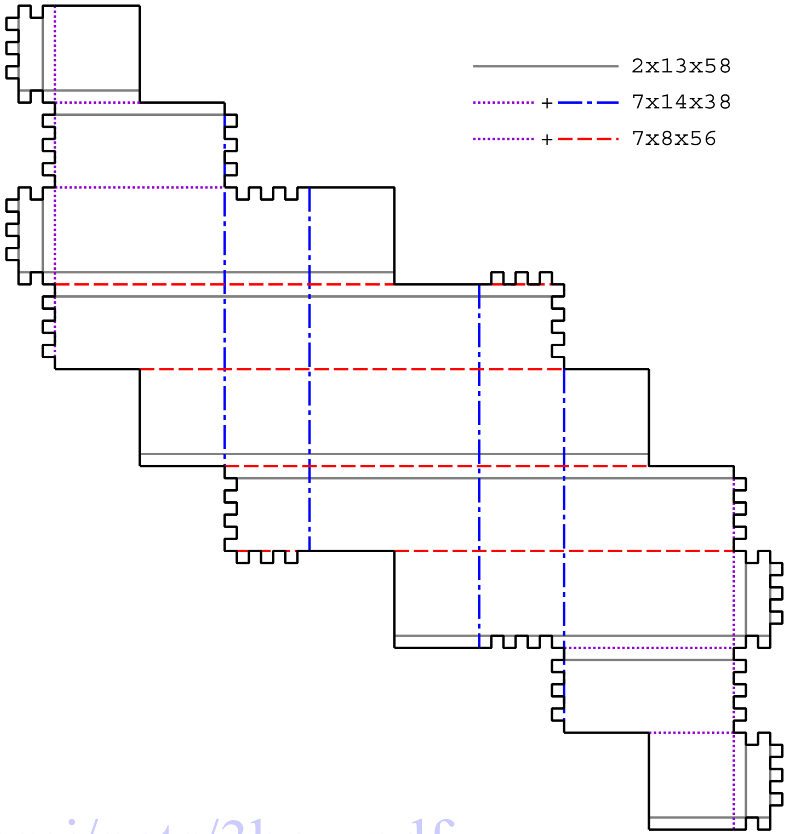
1×11×0 !

If you don’t like 1/2 , you can refine each square (□) into 4 squares (田)

# Finally: Common development of 3 boxes (1)

- February 2012, Shirakawa and Uehara finally found a common development of 3 boxes!!

[Basic idea] We fold one more box from a common development of 2 boxes in somehow….




You can find this pattern at
http://www.jaist.ac.jp/~uehara/etc/origami/nets/3box.pdf

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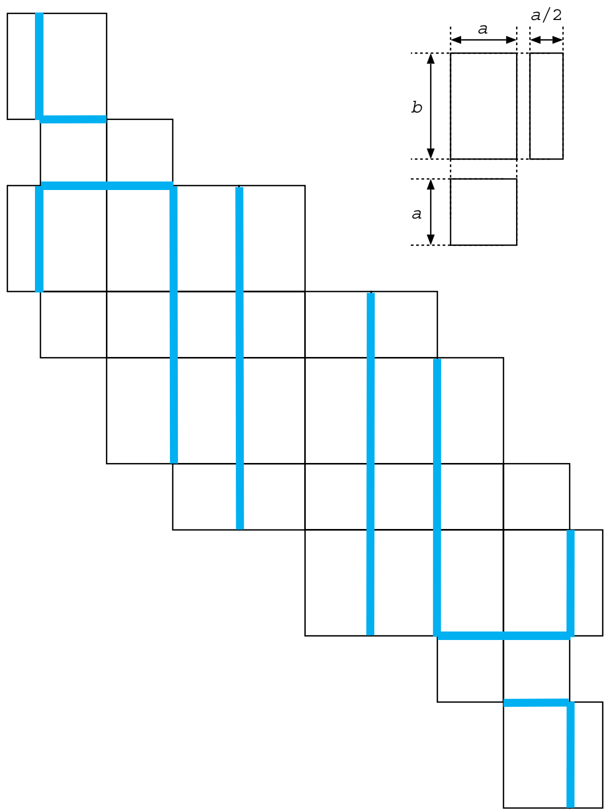


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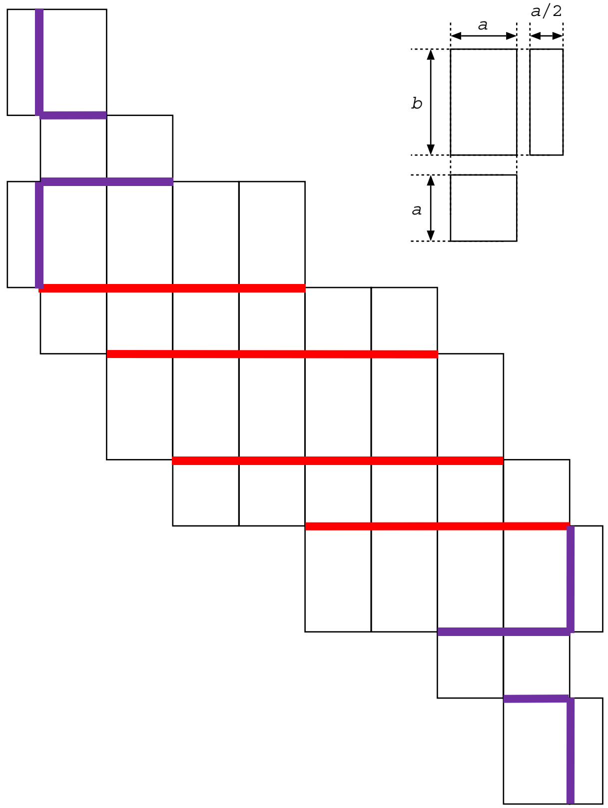


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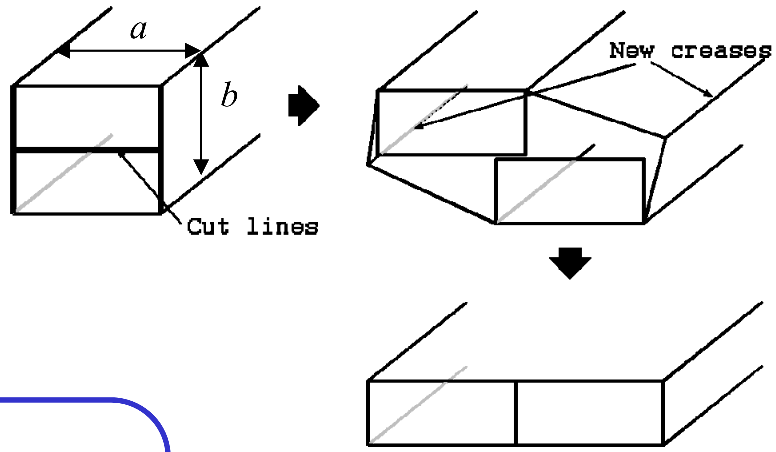


[No!!]

The idea works only when $a=2b$, which allow to translate from a rectangle of size $1\times2$ to a rectangle of size $2\times1$.

We may *squash* the box like this way?

You can find this pattern at
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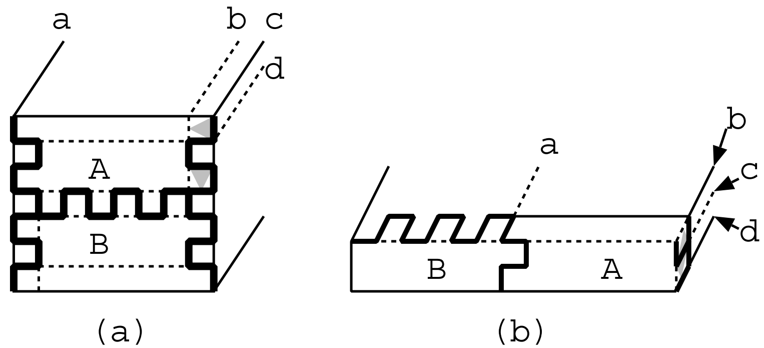


[Yes!!]
If we use a neat pattern!

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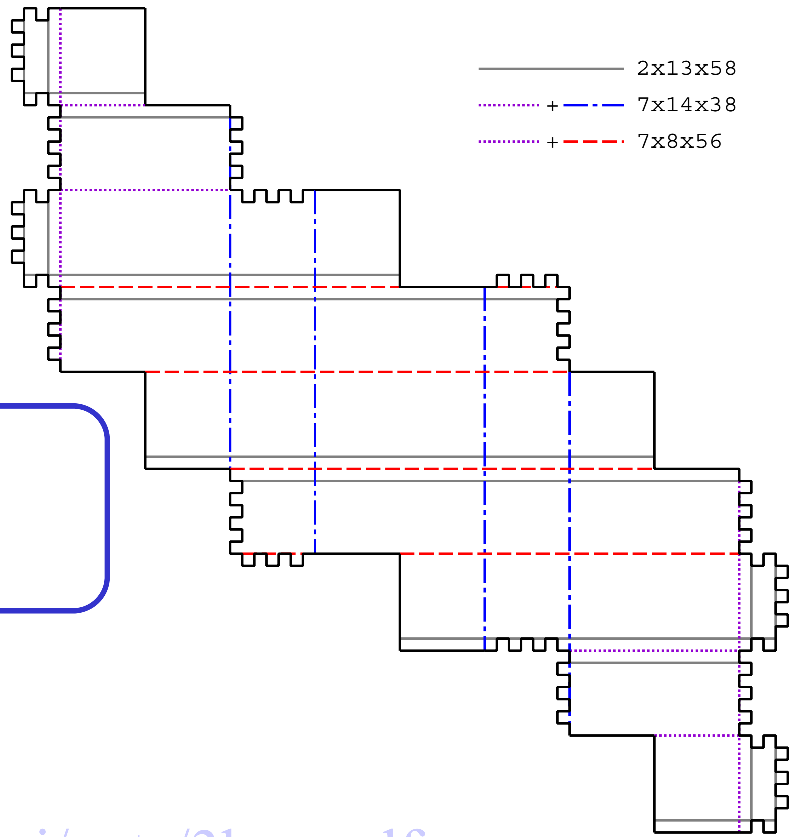


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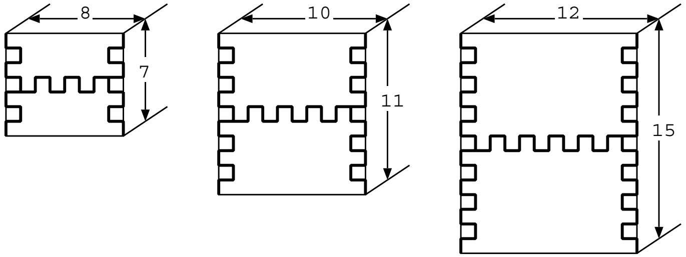



[Basic idea] We fold one more box from a common development of 2 boxes in somehow….

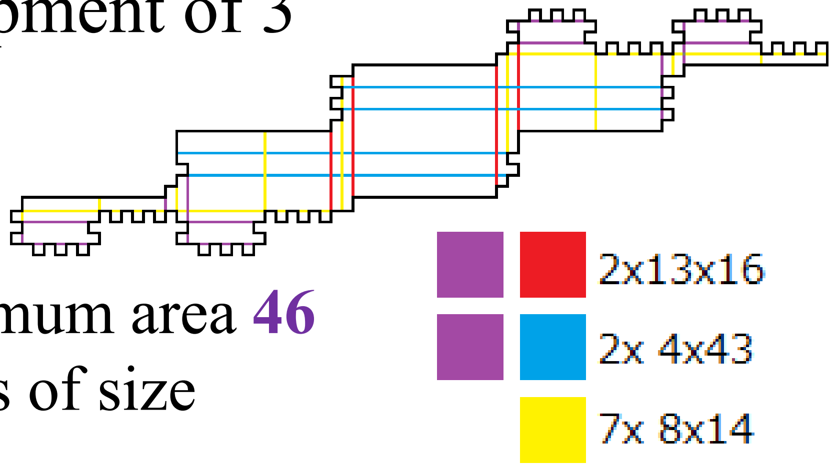
[Theorem]
There are infinitely many polygons that fold to three different boxes.

[Generalization]
- The base box has edges of flexible lengths.
- Zig-zag pattern can be generalized.

You can find this pattern at
http://www.jaist.ac.jp/~uehara/etc/origami/nets/3box.pdf

# Future work in those days

- The smallest common development of 3 boxes?



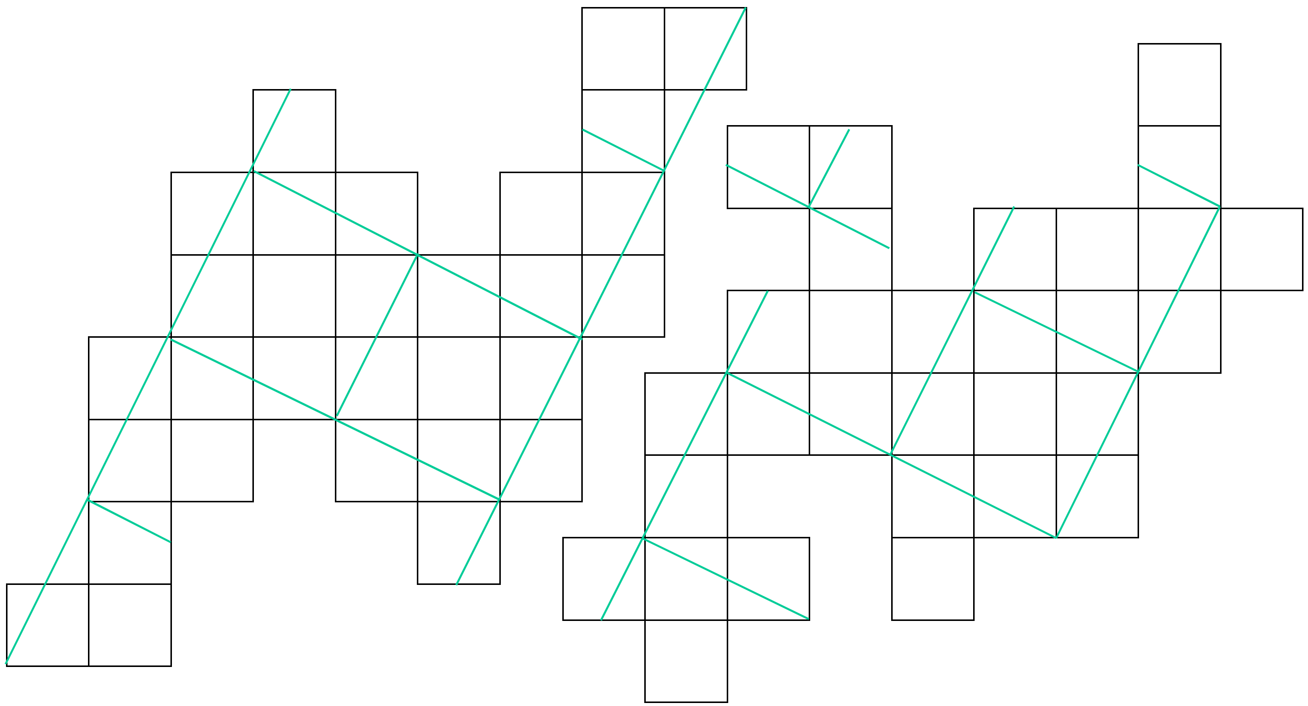

Using the idea, we obtain smallest one with **532 unit squares,** which is quite larger than the minimum area **46** that **may** allow us to fold 3 boxes of size $1\times1\times11$, $1\times2\times7$, $1\times3\times5$.

(Note: There are 2263 common developments of area **22** of two boxes of size $1\times1\times5$ and $1\times2\times3$.)

Are there common developments of 4 or more boxes?
(Is there any upper bound of this number?)

October 23, 2012: Email from Shirakawa…

"I found polygons of area 30 that fold to 2 boxes of size $1 \times 1 \times 7$ and $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. This area allows to fold of size $1 \times 3 \times 3$, it may be the smallest area of three boxes if you allow to fold along diagonal."



# Surface areas and possible size of boxes

If you want to find common developments of three boxes,

If you want to find common developments of four boxes,

| Area | 3-tuples | Area | 3-tuples |
|---|---|---|---|
| **<u>22</u>** | (1,1,5),(1,2,3) | 46 | (1,1,11),(1,2,7),(1,3,5) |
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Known results

In 2011, Matsui's program based on exponential time algorithm

- enumerated all developments of area 22
  - there are 2263 development of boxes of size 1×1×5 and 1×2×3
- ran in 10 hours on his desktop PC

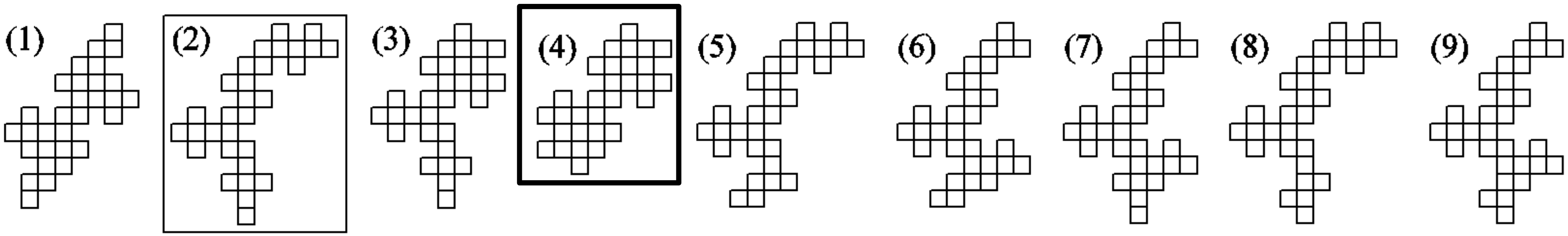
Area 30 was <u>on the edge</u>…

# My student, Dawei, succeeded! …on June, 2014, for his master thesis on September ;-)

- We completed enumeration of developments of area 30!
  [Xu, Horiyama, Shirakawa, Uehara 2015]

Note: Using BDD, the running time is reduced to 10 days!

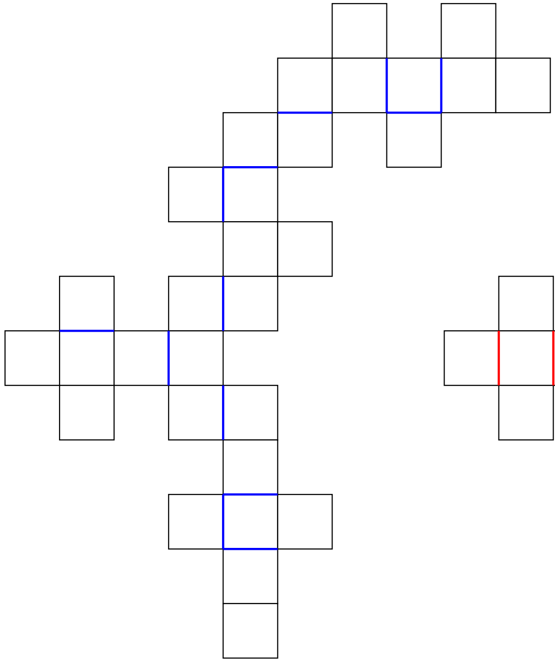
- Summary:
  - It took 2 months by Supercomputer (Cray XC 30) in JAIST.
  - There are 1080 common developments of 2 boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$
  - Among 1080, the following 9 can fold to a cube of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$.

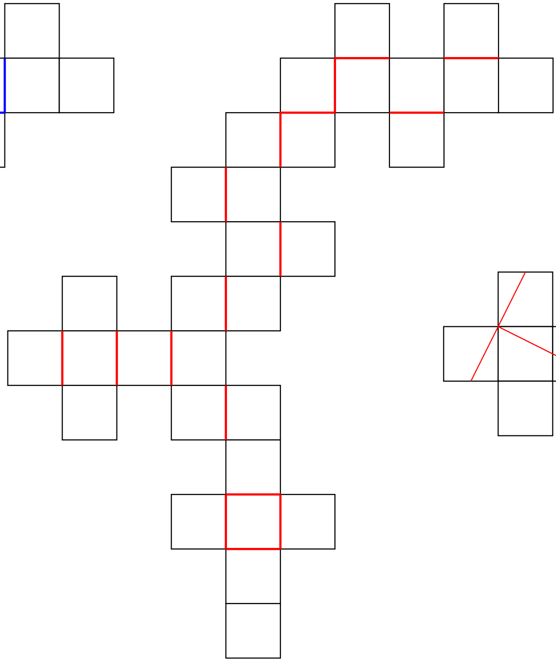


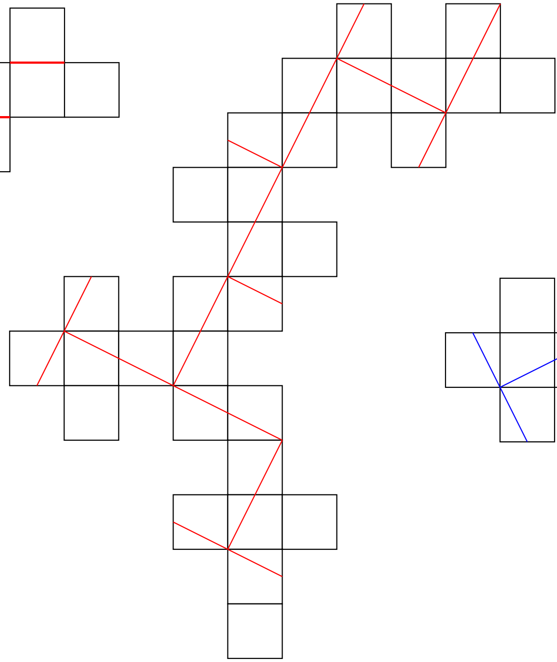

Quite surprisingly, (2) & (4) have 4 different ways for folding the boxes!!

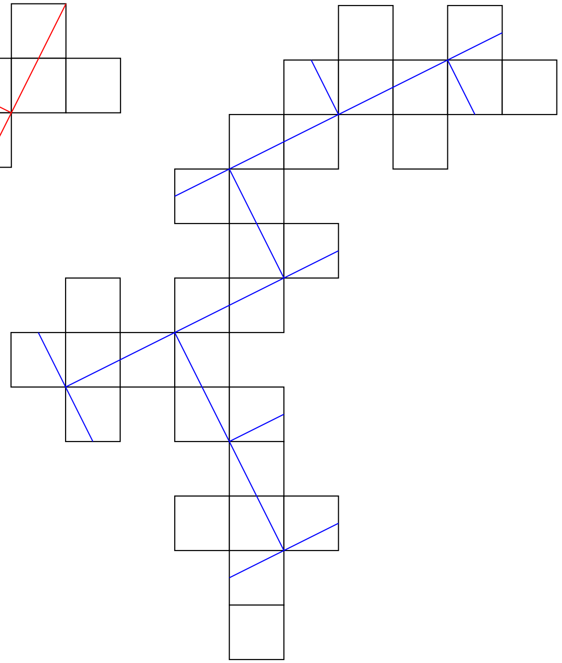
# Miracle Development

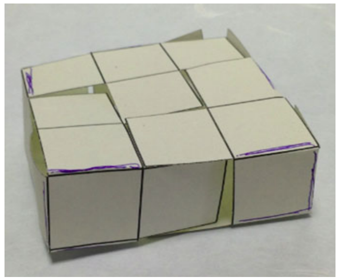
**This pattern has 4 ways of folding to box!!**



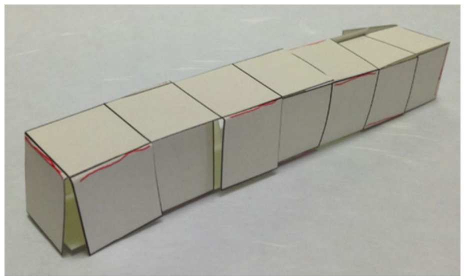




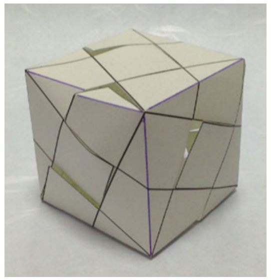




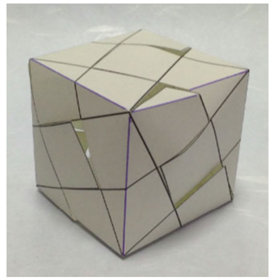
$1 \times 3 \times 3$



$1 \times 1 \times 7$



$\sqrt{5} \times \sqrt{5} \times \sqrt{5}$

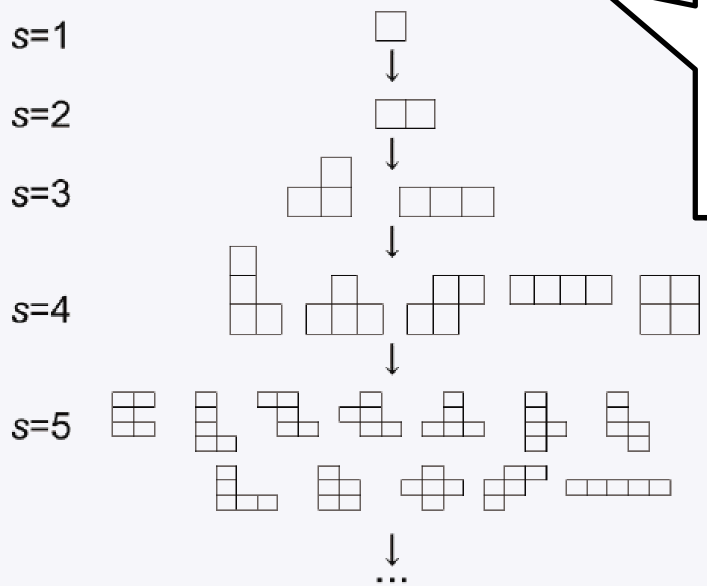


$\sqrt{5} \times \sqrt{5} \times \sqrt{5}$

# Brief Algorithm for finding them

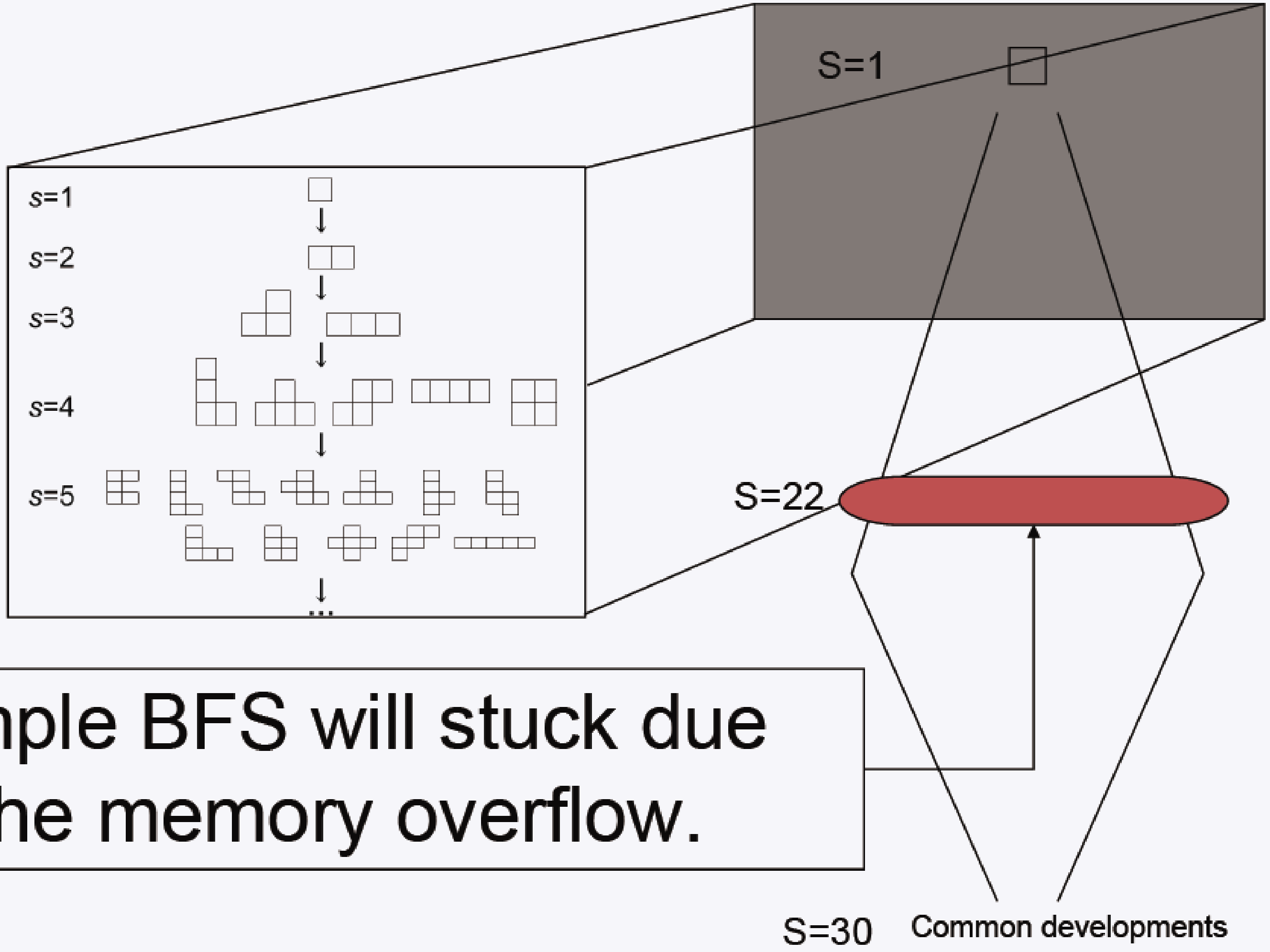
## The enumerate approach

- The basic idea is similar to finding two boxes of size 1×1×5 and 1×2×3 [6].
- We start from a single 1 square, then add another square adjacent to it, and extend the set of partial developments, repeat this step, untill 30 squares.




From Ph.D defense slides by Dawei on June 15, 2017

# The simple BFS gets stuck



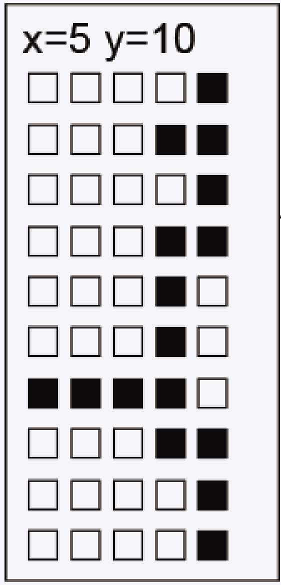

Simple BFS will stuck due to the memory overflow.

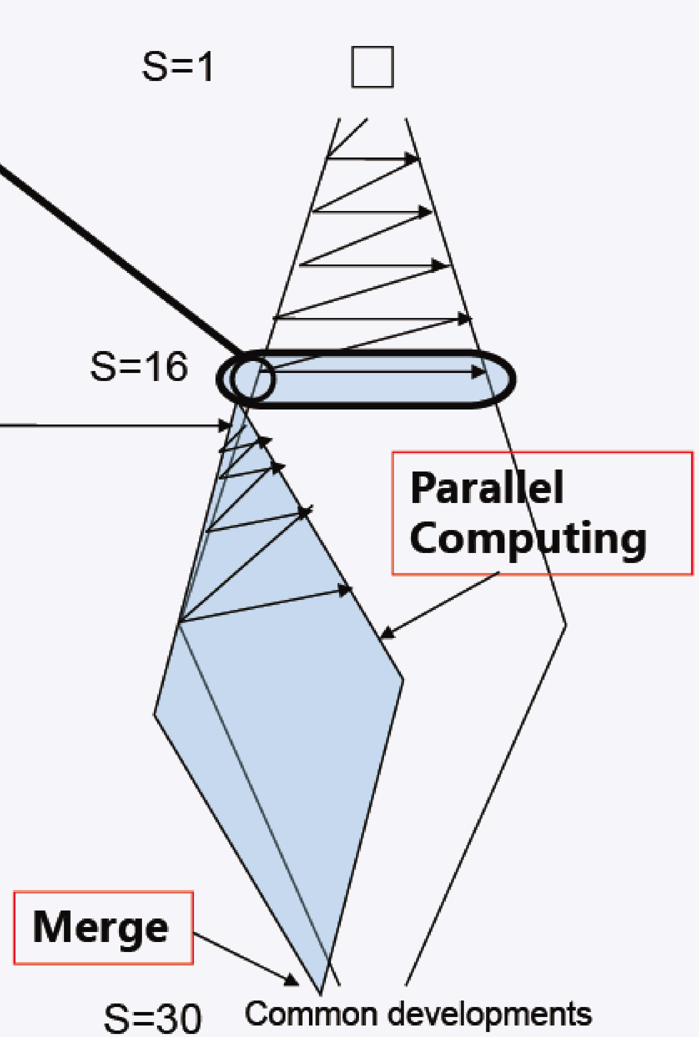
# Our solution

**Segmentation**

Step 16 generated 7486799 developments,
Divided them into 75 groups.

$development_0$,
$development_1$,
$development_2$,
.
.
.
.
.
.
.
.
$development_{7486798 / 75}$,



# Summary and future work…

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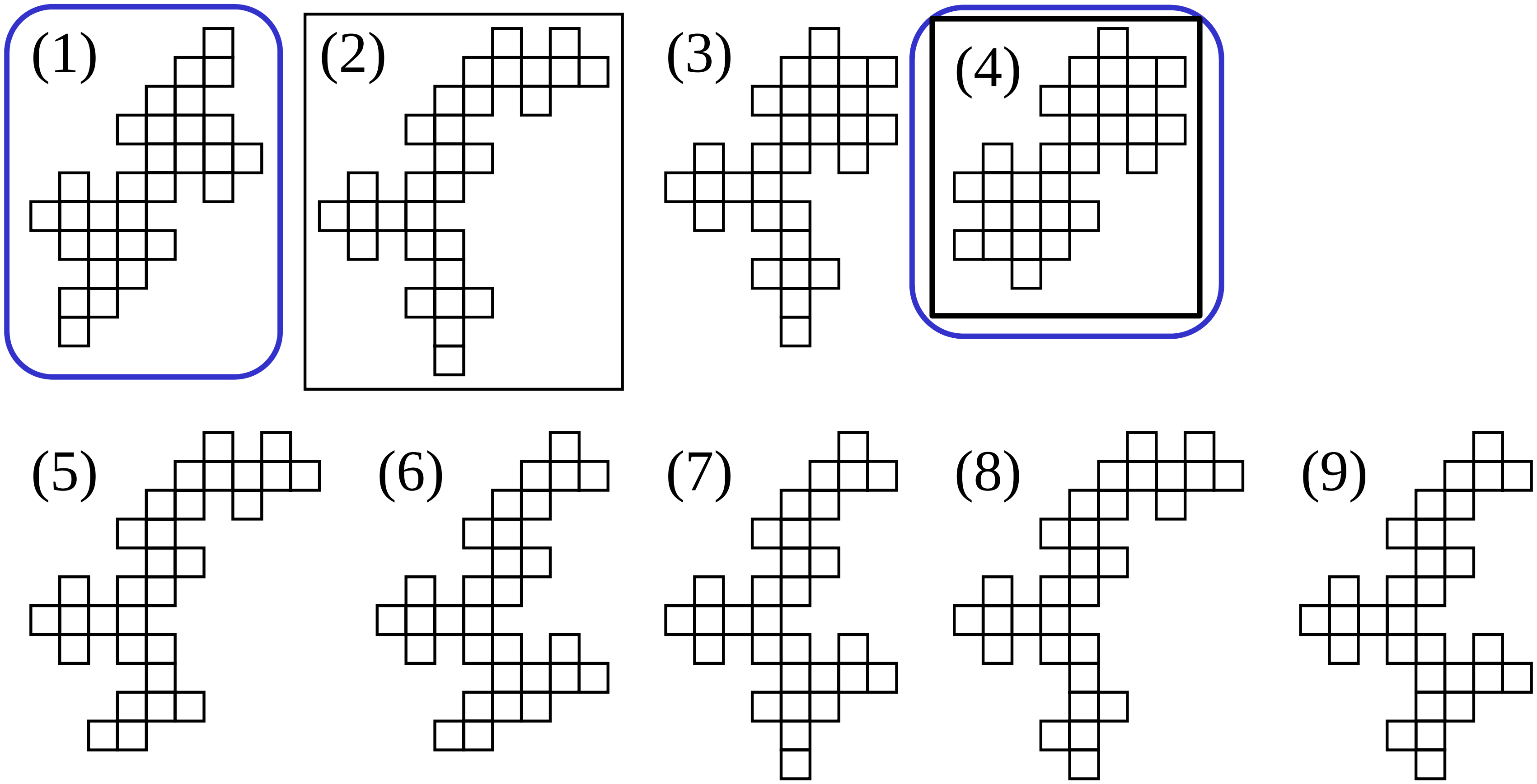
Known results

- In 2011, area 22 was enumerated in 10 hours on a desktop PC.
- In 2017, area 30 was enumerated in 2 months by a supercomputer, and improved to 10 days on a desktop PC.
- It seems to be quite hard to area 46 in this approach…

# Some progress…?

- We can try more on the symmetric ones…



# Some progress…?

- We can try more on the symmetric ones…
  1. The search space can be drastically reduced,
  2. Memory size is reduced into half, and
  3. Area can be incremented by 2.

(Quite sad) NEWS:

No common development of 3 boxes of areas 46 and 54

- Area 46: There are symmetric common developments of two different boxes of any pair of size $1 \times 1 \times 11$, $1 \times 2 \times 7$, and $1 \times 3 \times 5$, but there are no symmetric common development of 3 of them.
- Same as for the area 54 of size $1 \times 1 \times 13$, $1 \times 3 \times 6$, and $3 \times 3 \times 3$.

# Open problems

- Are there common developments of 3 boxes of size 46 or 54?
- Is there any common development of 4 boxes?
- Is there any upper bound of k of the number of boxes that share a common development? It is quite unlikely that there is a common development of 10,000 different boxes,,,?

FYI: The number of different polyominoes is known up to area 45. (by Shirakawa on OEIS)

# More open problems

The other variants of the following general problem:

For any polygon P, determine if you can fold to a (specific) convex polyhedron Q.

Known (related) results :

- General polygon P and convex polyhedron Q, there *is* a pseudo poly-time algorithm, however, …
  - It runs in $O(n^{456.5})$ time! (Kane, et al, 2009)
- When Q is a box, and polygon P,
  - Pseudo-poly-time algorithm for finding all boxes folded from P. [Mizunashi, Horiyama, Uehara 2019] (March, 2019)

There are many open problems, and young researchers had been solving them ☺

# じゃばらを高速に折る
# --- folding complexity ---

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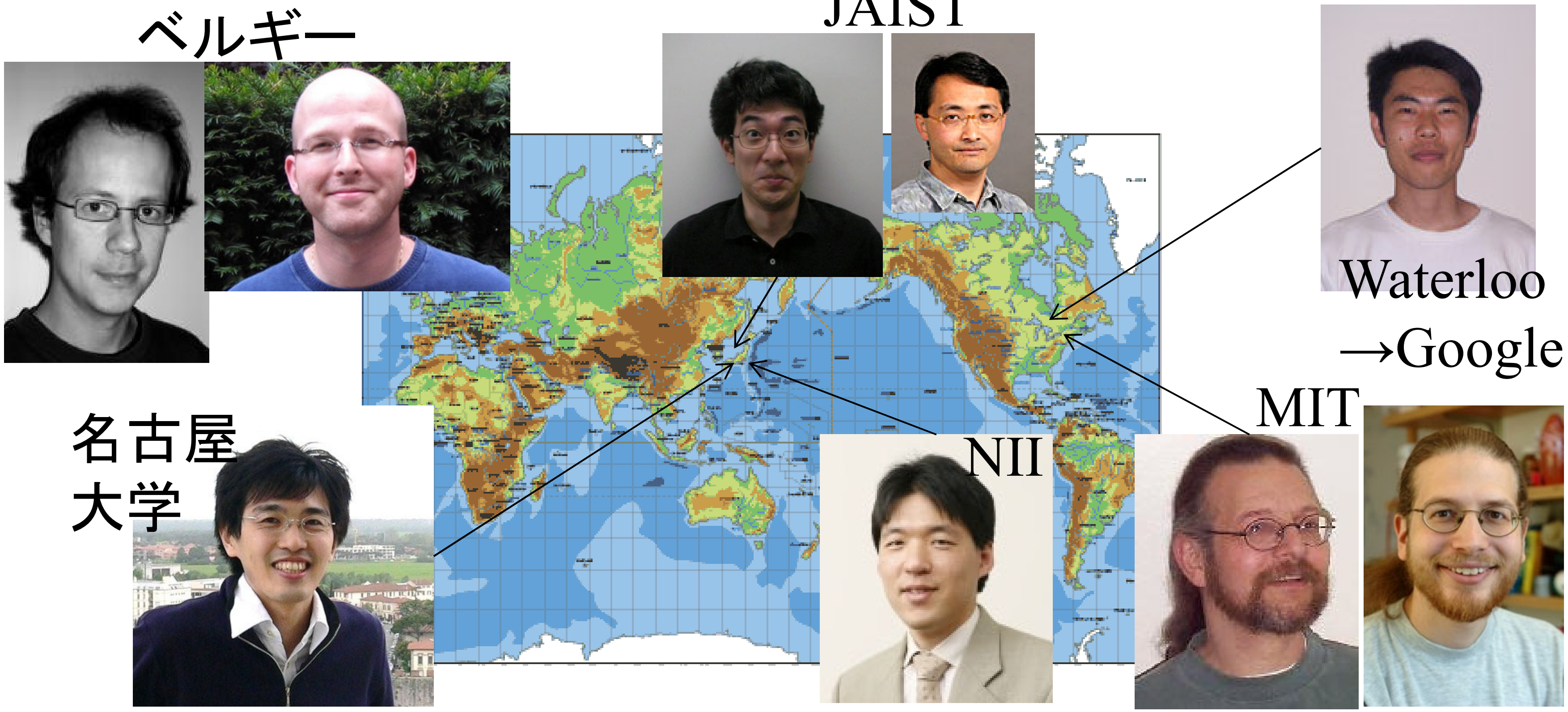
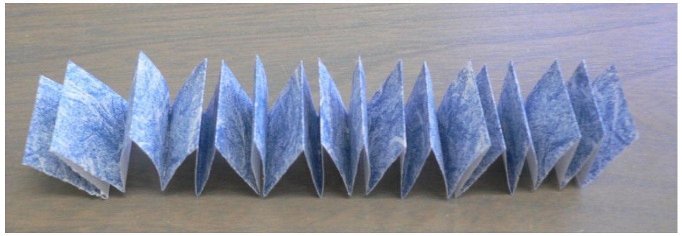
Goal:
折り紙の複雑さを時間計算量のアナロジーで評価
We estimate complexity of origami in a similar way of time complexity of computational complexity…

Journal Version:

J. Cardinal, E. D. Demaine, M. L. Demaine, S. Imahori, T. Ito, M. Kiyomi, S. Langerman, R. Uehara, and T. Uno: Algorithmic Folding Complexity, *Graphs and Combinatorics*, Vol. 27, pp. 341-351, 2011.



# Pleat folding is…




- Repeating of mountain and valley folding
- Basic operation in some origami
- Many applications

Special thanks to Takashi Hojo


http://km-sewing.seesaa.net/article/60694279.html

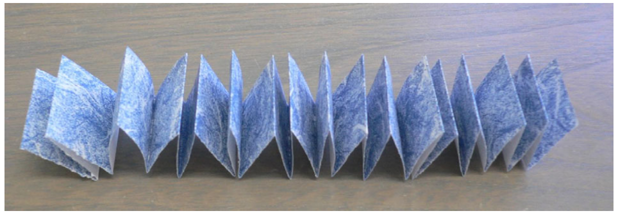

# Pleat fold[ing]

Repeating "folding in half" is the best way to make many creases.

- Pleat folding (in 1D)

- Naïve algorithm: $n$ time folding is a trivial solution
- We have to fold at least $\log n$ times to make $n$ creases

- More efficient ways…?
- General Mountain/Valley pattern?



- proposed at Open Problem Session on CCCG 2008 by R. Uehara.
- T. Ito, M. Kiyomi, S. Imahori, and R. Uehara: "*Complexity of pleats folding*", EuroCG 2009.

# • Complexity of Pleat Folding

Model :
Paper has 0 thickness

**[Main Motivation]** Do we have to make $n$ foldings to make a pleat folding with $n$ creases??

1. The answer is “No”!
   - ***Any pattern*** **can be made by** $\lfloor n/2 \rfloor + \lceil \log n \rceil$ **foldings**
2. Can we make a pleat folding in o($n$) foldings?
   - Yes!! …it can be folded in O($\log^2 n$) foldings.
3. Lower bound; log $n$
   - (We states $\Omega(\log^2 n / \log\log n)$ lower bound for pleat folding!!)

# • Complexity of Pleat Folding

**[Next Motivation]** What about general pleat folding problem for a given M/V pattern of length $n$?

- ***Any pattern*** **can be made by** $\lfloor n/2 \rfloor + \lceil \log n \rceil$ **foldings**

1. Upper bound:
   Any M/V pattern can be folded by $(4+\varepsilon)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right)$ foldings
2. Lower bound:
   Almost all mountain/valley patterns require $\frac{n}{3+\log n}$ foldings

[Note] Ordinary pleat folding is exceptionally easy pattern!

Difficulty/Interest come from two kinds of *Parities*:

- "Face/back" determined by layers
- Stackable points having the same parity

Input: Paper of length $n+1$ and a string $s$ in $\{M, V\}^n$

Output: Well-creased paper according to $s$ at regular intervals.

Basic operations

1. Flat {mountain/valley} fold {all/some} papers at an integer point (= simple folding)
2. Unfold {all/rewind/any} crease points (= reverse of simple foldings)

Rules

1. Each crease point remembers the last folded direction
2. Paper is rigid except those crease points

**Goal**: **Minimize the number of folding operations**

Note: We ignore the cost of unfoldings

# • Upper bound of Unit FP (1)

□ Any pattern can be made by $\lfloor n/2 \rfloor + \lceil \log n \rceil$ folding

1. *M*/*V* fold at center point according to the assignment
2. Check the center point of the folded paper, and count the number of *M*s and *V*s (we have to take care that odd depth papers are reversed)
3. *M*/*V* fold at center point taking majority
4. Repeat steps 2 and 3
5. Unfold all (cf. on any model)
6. Fix all incorrect crease points one by one

Steps 1~4 require log *n* and step 6 requires *n*/2 folding

□

# Upper bound of Pleat Folding(1)

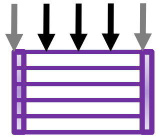
[Observation]
If $f(n)$ foldings achieve $n$ mountain foldings,
$n$ pleat foldings can be achieved by $2\,f(n/2)$ foldings.

The following strategy works;

- Make $f(n/2)$ mountain foldings at odd points;
- Reverse the paper;
- Make $f(n/2)$ mountain foldings at even points.

We will consider the
"**mountain folding problem**"

# Mountain folding in $\log^2 n$ folding



Step 1;

1. Fold in half until it becomes of length [vvv] (log *n*-2 foldings)
2. Mountain fold 3 times and obtain [MMM]
3. Unfold; vMMMvvvvvMMMvvvvvMMMvvvvvMMMvvvvv…

Step 2;

[MvvvvvM]

1. Fold in half until all "vvvvv"s are piled up (log *n*-3 foldings)
2. Mountain fold 5 times [MMMMMMM], and unfold
3. vMvMMMMMMMvMvvvvvMvMMMMMMMvMvvvvvMvM

Step 3; Repeat step 2 until just one "vvvvv" remains

vMvMMMvMMMvMMMMMMMvMMMvMMMvMvvvvvMvM

Step 4; Mountain fold all irregular vs step by step.

- #iterations of Steps 2~3; log *n*
- #valleys at step 4; log *n*

#foldings in total~ $(\log n)^2$

# • Lower bound of Unit FP

[Thm] Almost all patterns but $o(2^n)$ exceptions require $\Omega(n/\log n)$ foldings.

[Proof] A simple counting argument:

{surface/reverse} × {front/back}

– # patterns with $n$ creases $> 2^n/4 = 2^{n-2}$

– # patterns after $k$ foldings <

$$(2 \times n) \times (n+1) \times (2 \times n) \times (n+1) \times \ldots \times (n+1) \times (2 \times n)$$

M/V

Position

Possible unfolding

$$<(2n(n+1))^k$$

– We cannot fold most patterns after at most $k$ foldings if $\sum_{i=0}^{k}(2n(n+1))^i \leq (2n(n+1)+1)^k < 2^{n-2}$

– Letting $n \geq 2, k = O\left(\frac{n}{\log n}\right)$ we have $(2n(n+1)+1)^k = o(2^n)$ □

# Any pattern can be folded in $cn/\log n$ folding

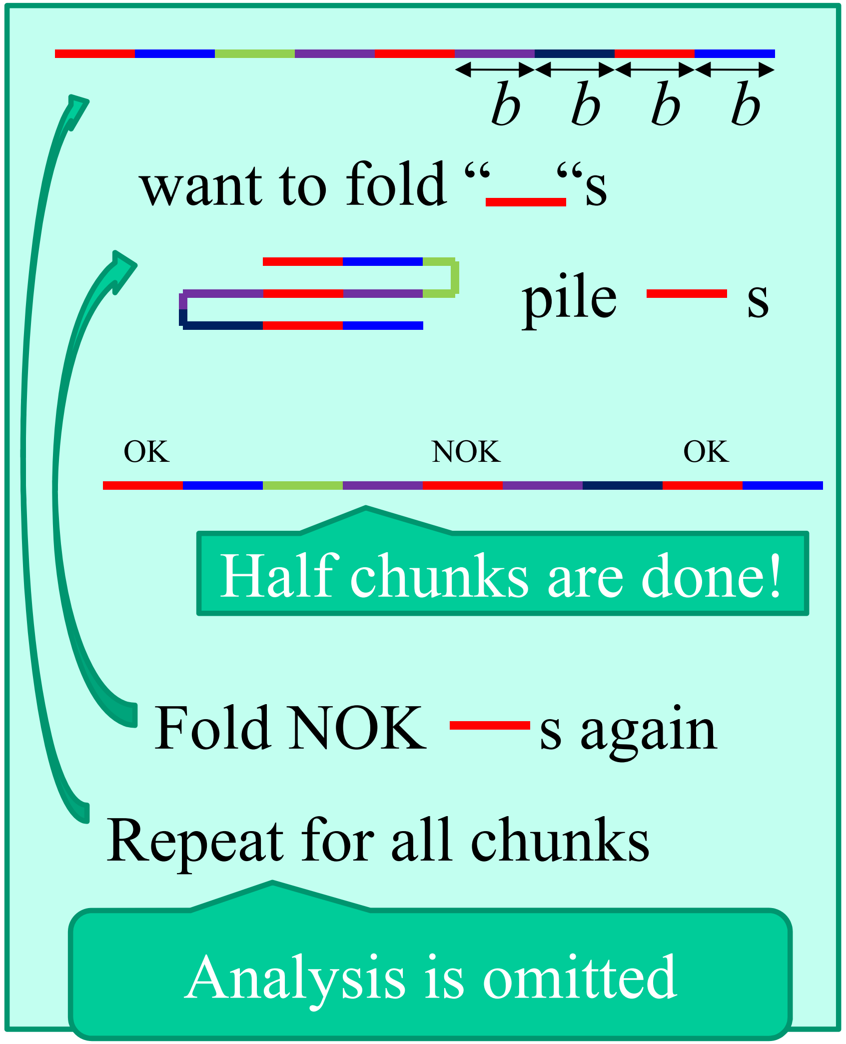
Select suitable $b$ depending on $n$.

Prelim.

- Split into *chunks* of size $b$;
  1. Each chunk is small and easy to fold
  2. #kinds of different $b$s are not so big

Main alg.

- For each possible $b$
  1. pile the chunks of pattern $b$ and mountain fold them
  2. fix the reverse chunks
  3. fix the boundaries



# • Open Problems

- Pleat foldings
  - Make upper bound $O(\log^2 n)$ and lower bound $\Omega(\log^2 n / \underline{\log\log n})$ closer
- "**Almost all** patterns are difficult", but…
  - **No explicit M/V pattern** that requires ($cn/\log n$) folding
- When "unfolding cost" is counted in…
  - Minimize #folding + #unfolding