I238 計算の理論

上原 隆平

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I238 Computation Theory

by

Prof. Ryuhei Uehara

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計算量の理論

- ゴール1:
  - “計算可能な*関数/問題/言語/集合*”
    - 関数には2種類存在する;
      1. 計算不能(!)な関数
      2. 計算可能な関数
- ゴール2:
  - 「問題の困難さ」を示す方法を学ぶ
    - 計算可能な問題であっても、*手におえない*場合がある！
      - 計算に必要な資源(時間・領域)が多すぎるとき

Computational Complexity

- Goal 1:
  - "*Computable Function/Problem/Language/Set*"
    - We have two functions;
      1. Functions that are not computable!
      2. Functions that are computable.
- Goal 2:
  - How can you show "*Difficulty of Problem*"
    - There are *intractable* problems even if they are computable!
      - because they require too many resources (time/space)!

計算量の理論

- ゴール1:
  - “計算可能な*関数/問題/言語/集合*”
    - 関連する専門用語;
      計算可能性、対角線論法

グラフ理論/グラフの問題/グラフアルゴリズム
- グラフ理論超入門
  - 無向グラフ，有向グラフ，多重グラフ，木構造、平面グラフ
- グラフ上の問題とアルゴリズム
  - 探索問題：オイラー閉路、最小全域木など

- ゴール2:
  - 「問題の困難さ」を示す方法を学ぶ
    - 関連する専門用語;
      クラスNP, P≠NP予想, NP困難性, 多項式時間還元

Computational Complexity

- Goal 1:
  - "*Computable Function/Problem/Language/Set*"
    - Technical terms;
      computability, diagonalization

- Goal 2:
  - How can you show "*Difficulty of Problem*"
    - Technical terms;
      The class NP, P≠NP conjecture, NP-hardness, polynomial time reduction

Graph Theory/Graph Problems/
Graph Algorithms

- Introduction to Graph Theory
  - (Un)directed graph, multi-graph, tree, planar graph
- Graph problems and graph algorithms
  - Search Problems: Euler cycle, minimum spanning tree

中間テストの結果:

- (そういえば)レポート1の平均点: 14.6/20
- 中間テストの平均点:16.77/30

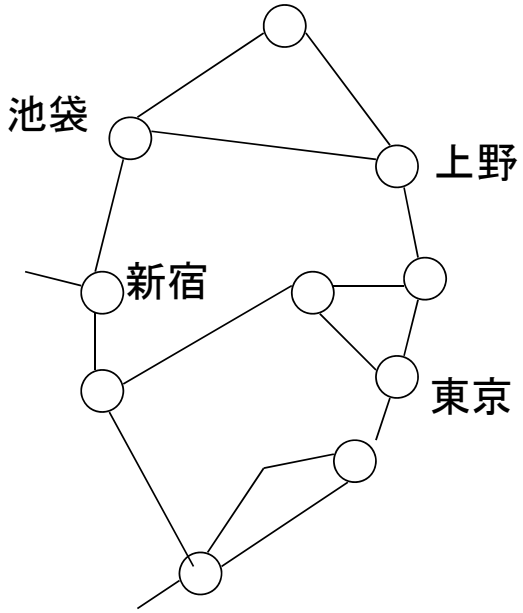
点数	0-5	6-10	11-15	16-20	20-25	26-30
人数	7	2	10	1	4	11

  - 合計すると, これまでのところの平均点:31.37/50＝62.74/100
- 結果を知りたい人は:
  - 必ず今週中にメールください.(特に上原は返事を出さないけど)
  - 来週の月曜日に一斉にメールで返します.(コメントはあったりなかったり?)
  - メールを出したのに返事がない人は, 問い合わせを. . .
- レポートの解答と解説: いま, やります.

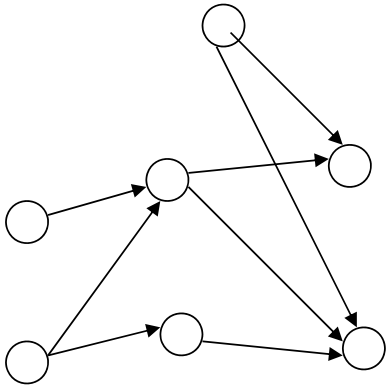
グラフ超入門

- 頂点を辺で結んだもの
  - 無向グラフ: 辺に方向がない
  - 有向グラフ: 辺に方向がある

例: 路線図



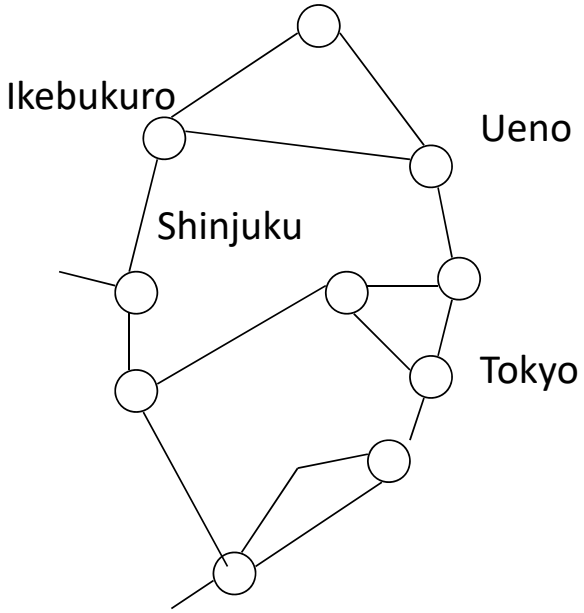
例: 講義の履修順序



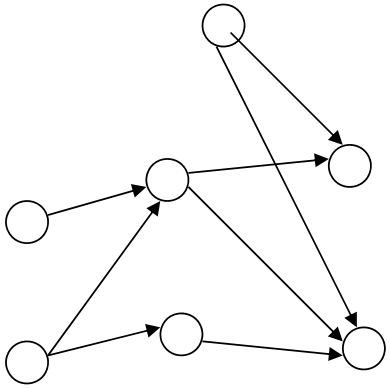
Super Introduction to Graph Theory

- A graph consists of a set of vertices joined by edges
  - (Undirected) graph: each edge has no direction
  - Directed graph: an edge has a direction

Ex: Railways



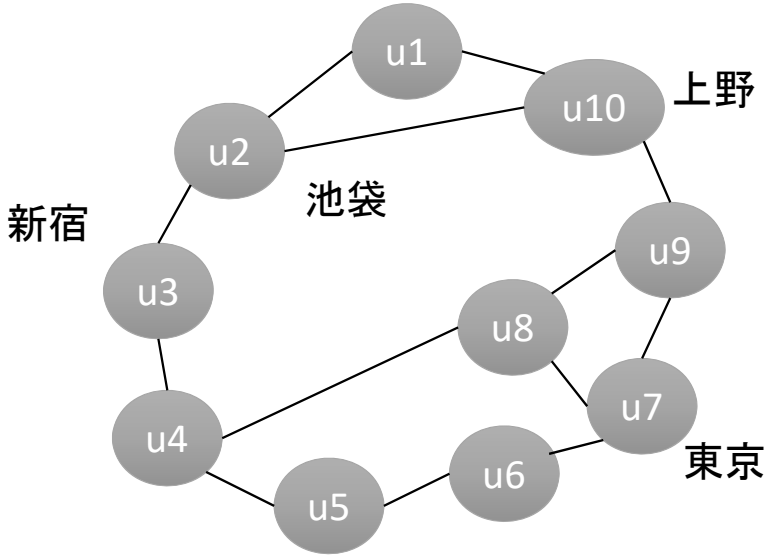
Ex: Relationship between courses



グラフ: 表記

- グラフ: $G = (V, E)$
  - V: 頂点集合，E: 辺集合
- 頂点: $u, v, \ldots \in V$
- 辺: $e = \{u, v\} \in E$（無向辺）
  $a = (u, v) \in E$（有向辺）
- 頂点や辺は重みを持つことも
  - $w(u), w(e)$
  - 距離，金額，時間など
- 頂点の次数
  - 無向グラフの場合
    頂点vにつながっている辺の数
    deg(v)と書く
  - 有向グラフの場合
    入次数/出次数を区別する
    indeg(v), outdeg(v)などと書く

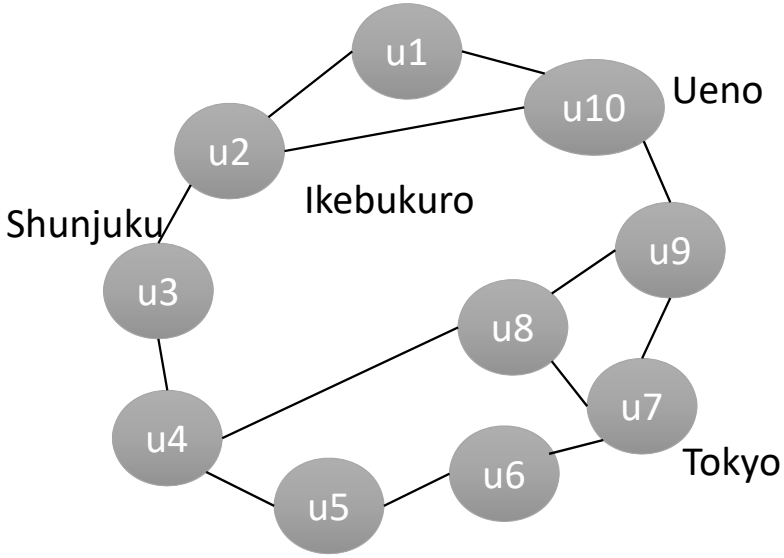
なぜか |V|=n, |E|=m
と書く「常識」がある



Graph: Notation

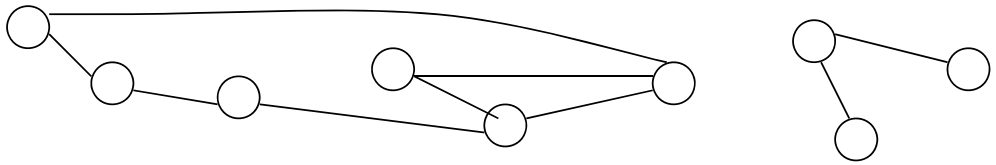
- Graph: $G = (V, E)$
  - V: Vertex set, E: Edge set
- Vertex: $u, v, \ldots \in V$
- Edge: $e = \{u, v\} \in E$ (undirected)
  $a = (u, v) \in E$ (directed)
- Sometimes, vertices and edges may have "weights"
  - $w(u)$, $w(e)$
  - Distance, cost, time, etc.
- Degree of a vertex
  - Undirected graph:
    The number of edges incident to a vertex v denoted by deg(v)
  - Directed graph:
    Indegree and Outdegree are distinguished denoted by, e.g., indeg(v), outdeg(v)

It is common to denote by
|V|=n, |E|=m.

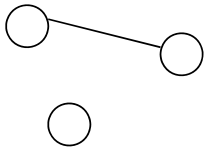


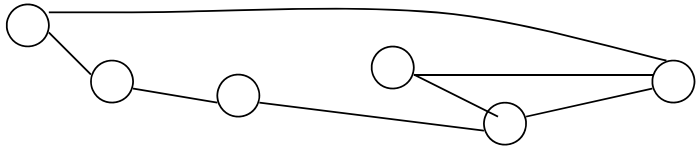
グラフ: 基本的な用語 (1/2)

- 路 (path): 辺で隣り合う頂点を繋いだもの
  - 単純路 (simple path): 同じ頂点を複数回通らない



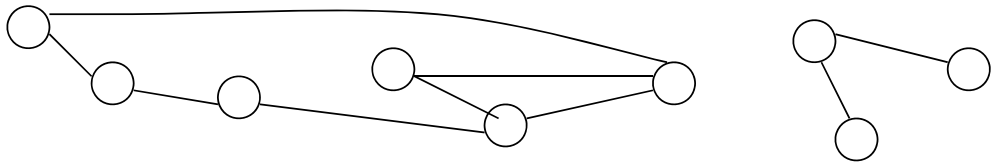
- 閉路 (cycle, closed path): *v* から *v* への路
- 連結グラフ (connected graph): どの2頂点間にも路が存在するグラフ



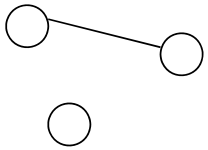


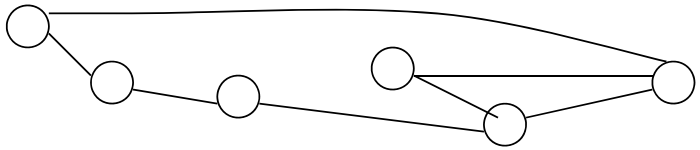
Graph: Basic terms (1/2)

- Path: a sequence of vertices joined by edges
  - Simple path: It never visit the same vertex twice or more



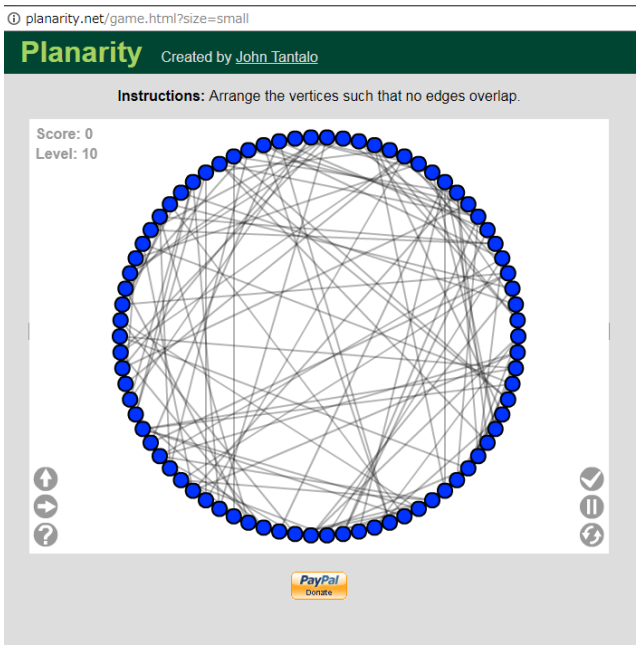
- Cycle, closed path: a simple path from *v* to *v*
- Connected graph: graph s.t. any two vertices are joined by a path



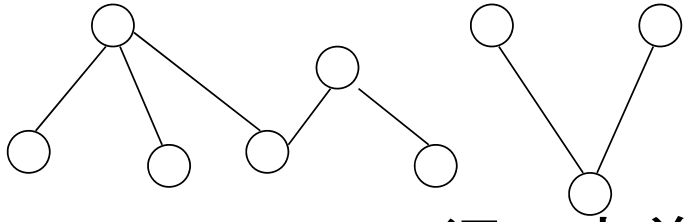


グラフ: 基本的な用語 (2/2)

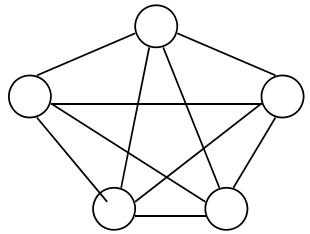
http://planarity.net/ より
（かなりハマるので危険）



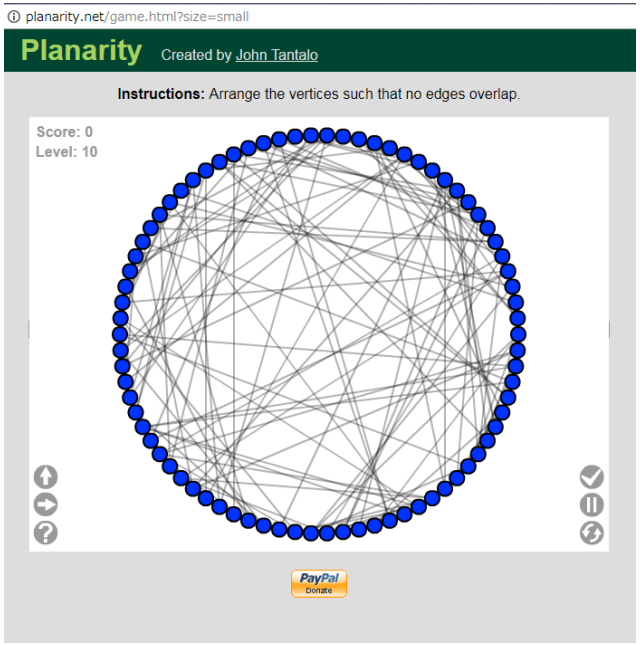
- 森 (forest): 閉路を含まないグラフ
- 木 (tree): 連結で閉路を含まないグラフ



- 平面的グラフ (planar graph): 辺の交差なしで平面に描画できるグラフ
- （平面グラフ(plane graph) : 描画情報つきの平面グラフ）
- 完全グラフ (complete graph): 全ての頂点対を辺で隣接させたグラフ
  - 完全グラフ K_5 は極小非平面的グラフ
    （証明は難しい : やってみたけどできなかったでは証明にならない）

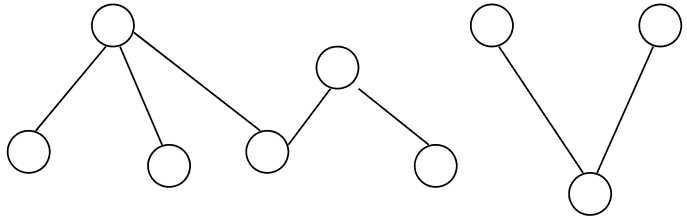


http://planarity.net/
(It is dangerous for students ☺)

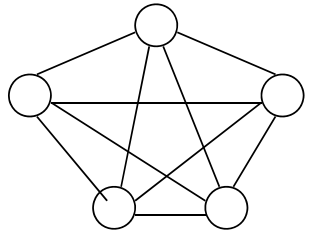


Graph: Basic terms (2/2)

- Forest: graph having no cycle (or acyclic)
- Tree: connected acyclic graph



- Planar graph: a graph drawable with no crossing of edges
- (Plane graph: Planar graph with plane drawing)
- Complete graph: graph s.t. all pairs are joined by edges
  - A complete graph K_5 is a minimal non-planar graph



グラフの基本的な性質

練習問題:
握手補題の有向
グラフ版を考えよ

定理（握手補題）: 任意の無向グラフ G=(V,E) について,

$$\sum \deg(v) = 2|E|$$

[証明] それぞれの辺は，両端点で，上記の式の左辺に2ずつ貢献するから.

定理: 頂点数 n の木の辺の数は n-1
[証明] 略.

練習問題:
木には葉（次数が1の頂点）が
必ず存在することを証明せよ

定理: 頂点数 n の平面グラフの辺の数は高々 3n-6
[証明] 略.

Basic properties of graphs

Exercise:
Consider directed version

Theorem (Handshake lemma): For any undirected graph G=(V,E),

$$\sum \deg(v) = 2|E|$$

[Proof] Each edge contributes twice to the left hand at its both endpoints.

Theorem: The number of edges of a tree of n vertices is n-1
[Proof] Omitted.

Theorem: The number of edges of a planar graph of n vertices is at most 3n-6
[Proof] Omitted.

Exercise:
Prove that there exists at least one leaf (vertex of degree 1) in any tree.

グラフアルゴリズムの計算量

- 頂点数 n, 辺数 $m \in O(n^2)$
  - 無向グラフ: $m \leqq n(n-1)/2$
  - 有向グラフ: $m \leqq n(n-1)$

O()記法は，近々やります．

- 平面グラフでは $m \in O(n)$

- グラフアルゴリズムの計算量は
  n や m の式で表す

Computational complexity of graph algorithms

- n vertices, m edges, $m \in O(n^2)$
  - Undirected graph: $m \leqq n(n-1)/2$
  - Directed graph: $m \leqq n(n-1)$

O-notations will be taught soon.

- On a planar graph, $m \in O(n)$

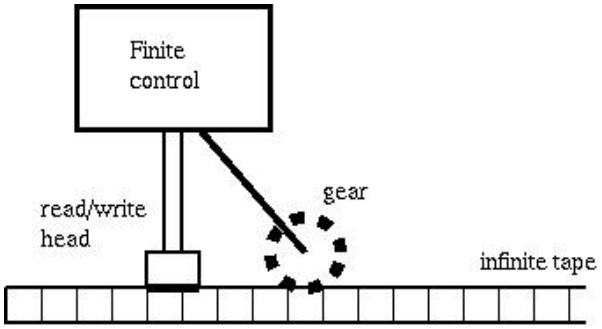
- Computational complexity of a graph algorithm is described by a function of n and m.

補足：計算モデルの話

- この講義の中ではこうした「些細な」違いは問題にならない．
- こうした違いを超えられる議論方法を学ぶ

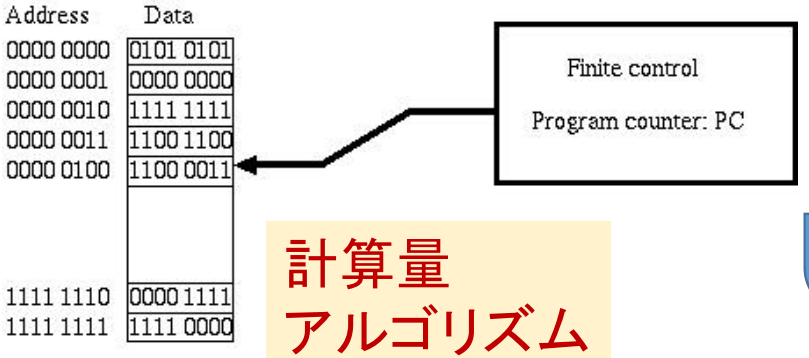
- チューリング機械⇔RAMモデル⇔実際のコンピュータ

チューリング機械



計算の理論
計算量

RAMモデル

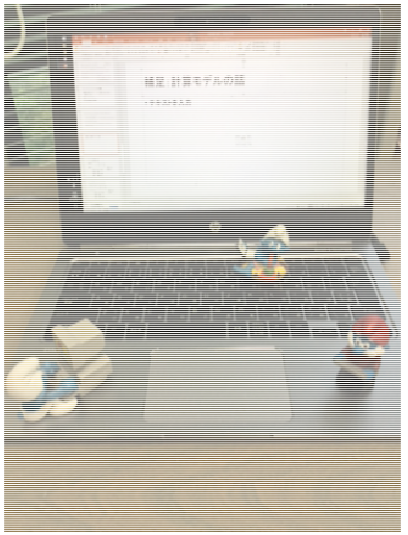


計算量
アルゴリズム

ランダムアクセス性がある
基本演算が多様

入出力は
考慮しない

実際のPC

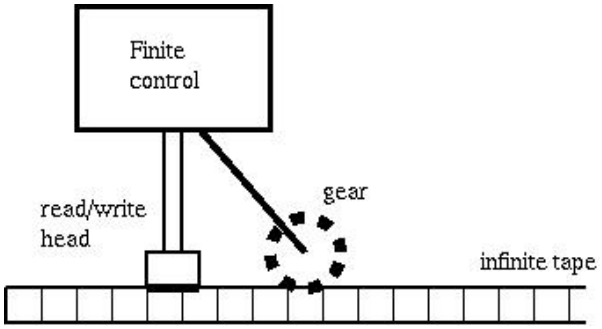


Suppl：Computation models

- Such difference is “trivial” issue in this course.
- We will learn how to deal it soon.

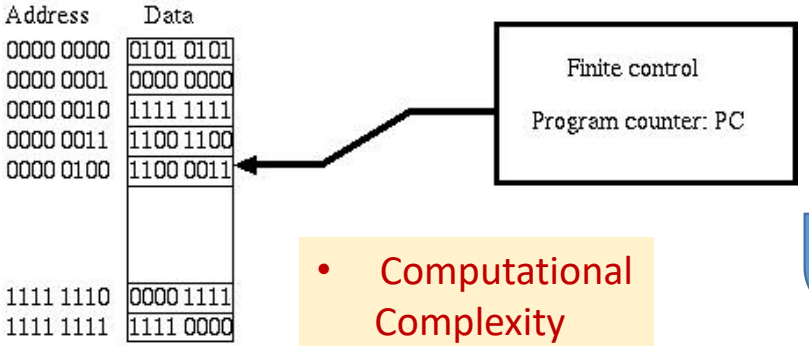
- Turing Machine⇔RAM Model⇔Real Computer

Turing Machine



- Computation
- Computational complexity

RAM Model

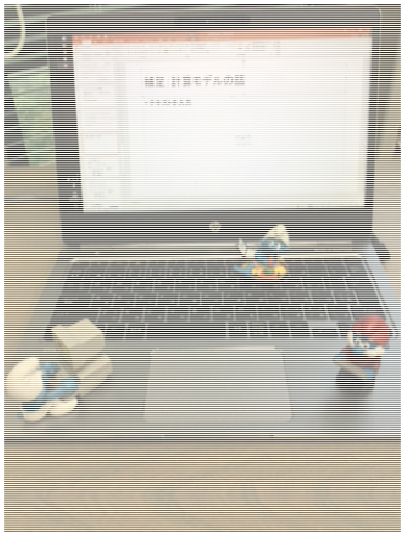


Random Access
Various of basic operations

- Computational Complexity
- Algorithm

Input/Output are not issue.

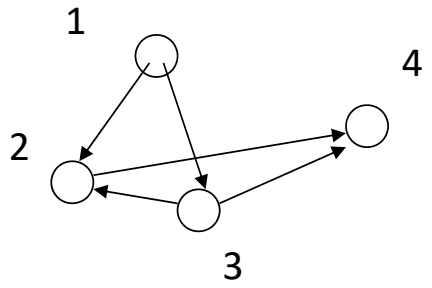
Real PC



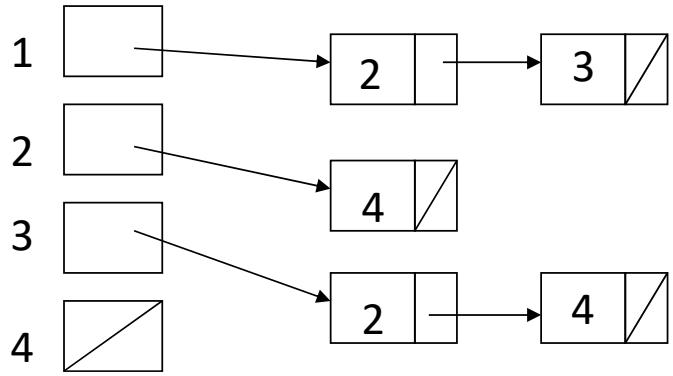
グラフの表現方法

- 隣接行列

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



- 隣接リスト

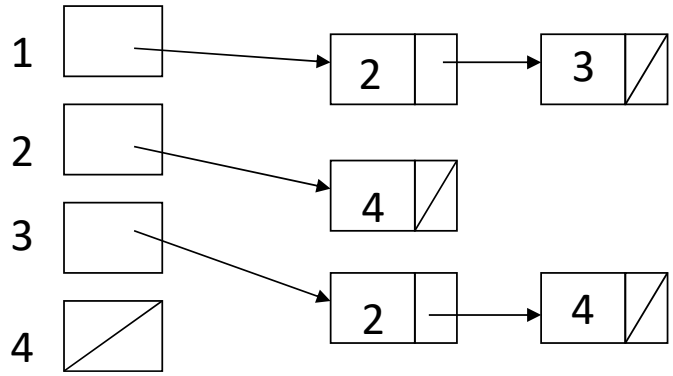


Representations of a graph

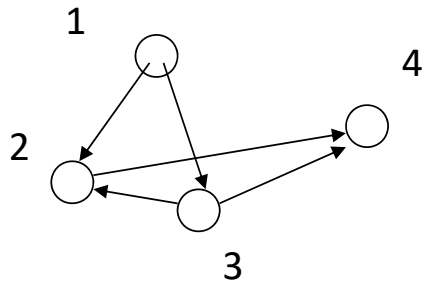
- Adjacency matrix

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Adjacency list



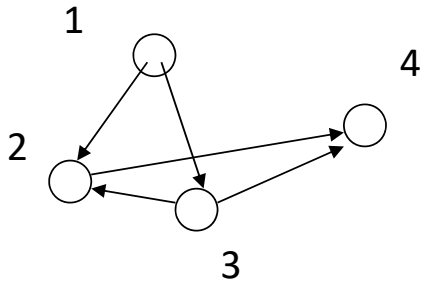
vertices Linked list of adjacent vertices



グラフの表現方法: 行列表現(隣接行列)

- $(u, v) \in E \Rightarrow M[u, v] = 1$
- $(u, v) \notin E \Rightarrow M[u, v] = 0$

辺重み付きグラフに拡張するのは簡単

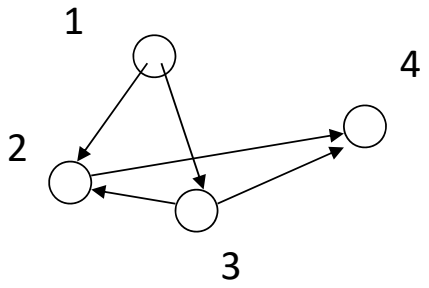


$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Representation of a graph: Matrix representation (adjacency matrix)

- $(u, v) \in E \Rightarrow M[u, v] = 1$
- $(u, v) \notin E \Rightarrow M[u, v] = 0$

It is easy to extend to edge-weighted graph.



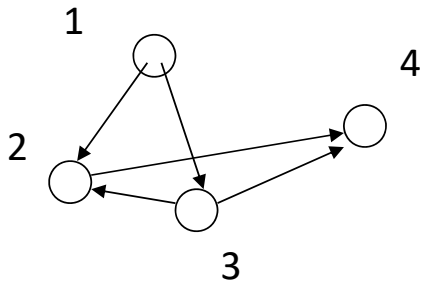
$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

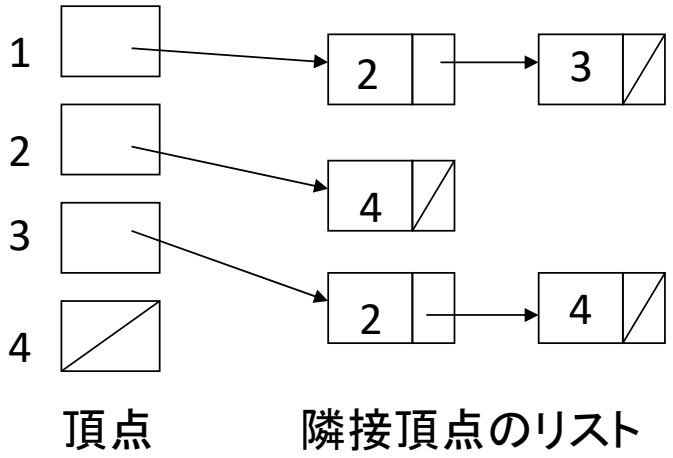
グラフの表現方法: リスト表現(隣接リスト)

- $(u, v) \in E \Leftrightarrow v \in T(u)$
  - $T(u)$ は u の隣接点のリスト

頂点重み付きグラフに拡張するのは簡単

- 例:



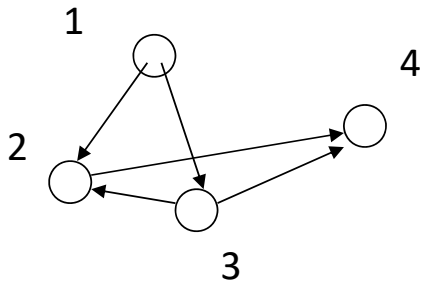


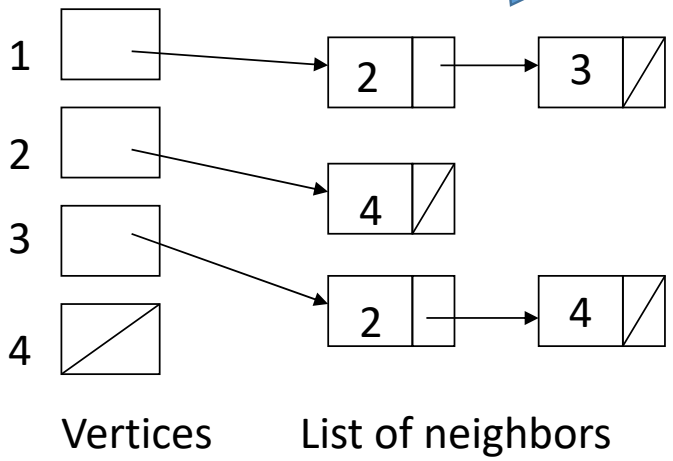
Representation of a graph: List representation (adjacency list)

- $(u, v) \in E \Leftrightarrow v \in T(u)$
  - $T(u)$ is the list of neighbors of u

It is easy to extend to vertex-weighted graph.

- Example:





隣接行列 vs 隣接リスト

- メモリ量
  - 隣接行列: $\Theta(n^2)$
  - 隣接リスト: $\Theta(m \log n)$
- 隣接チェックにかかる時間: $(u, v) \in E$?
  - 隣接行列: $\Theta(1)$
  - 隣接リスト: $\Theta(n)$

Q. グラフの更新 (e.g., 頂点・辺の追加・削除) は？

Adjacency matrix v.s. Adjacency list

- Memory
  - Adjacency matrix: $\Theta(n^2)$
  - Adjacency list: $\Theta(m \log n)$
- Time to check if $(u, v) \in E$?
  - Adjacency matrix: $\Theta(1)$
  - Adjacency list: $\Theta(n)$

Q. How about update of the graph?
(e.g., Add/remove of a vertex/edge)

ミニ演習(Some exercises)

- 頂点数4の無向グラフを全て描け(Draw all undirected graphs of 4 vertices)
  - 頂点にラベルはないものとする(Each vertex has no label)
  - 描き方を変えれば同じになるものは省く(“Same” graphs should be reduced)
    - cf. グラフの同型性(Graph Isomorphism)

- 頂点数 *n* の無向グラフは高々 2^{n^2} 個しかないことを証明せよ
  (Prove the number of undirected graphs of n vertices is at most 2^{n^2}.)