A Categorical Semantics for Polarized MALL

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Abstract

In this paper, we present a categorical model for Multiplicative Additive Polarized Linear Logic MALLP, which is the linear fragment (without structural rules) of Olivier Laurent's Polarized Linear Logic. Our model is based on an adjunction between reflective/coreflective full subcategories $\mathcal{C}_-/\mathcal{C}_+$ of an ambient $*$-autonomous category $\mathcal{C}$ (with products). Similar structures were first introduced by M. Barr in the late 1970's in abstract duality theory and more recently in work on game semantics for linear logic. The paper has two goals: to discuss concrete models and to present various completeness theorems.

As concrete examples, we present (i) a hypercoherence model, using Ehrhard's hereditary/anti-hereditary objects, (ii) a Chu-space model, (iii) a double gluing model over our categorical framework, and (iv) a model based on iterated double gluing over a $*$-autonomous category.

For the multiplicative fragment MLLP of MALLP, we present both weakly full (Läuchli-style) as well as full completeness theorems, using a polarized version of functorial polymorphism in a double-glued hypercoherence model. For the latter, we introduce a notion of polarized $\uparrow$-softness which is a variation of Joyal's softness. This permits us to reduce the problem of polarized multiplicative full completeness to the nonpolarized MLL case, which we resolve by familiar functorial methods originating with Loader, Hyland, and Tan. Using a polarized Gustave function, we show that full completeness for MALLP fails for this model.

Contents

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1 Introduction

Girard [20] introduced the notion of "polarized" (positive and negative) formulas in his study of the theory LC, a "constructive" version of classical logic based on linear logic. These polarities turn out to be related to the notion of *focussing* in linear logic proof search, a method introduced by J-M Andreoli [4, 5, 38]. In a related direction, many papers in Game Semantics for linear logic have also stressed the notion of polarities, beginning with Lamarche [32] (see also the survey [1]).

Olivier Laurent [34, 35] began a systematic study of polarized versions of linear logic. In his thesis he introduced polarized proof structures and nets which are simpler than the non-polarized original versions. He introduced many interesting proof-theoretical and semantical techniques and results for polarized logics, notably for full polarized linear logic LLP. In this theory, the exponentials are polarity changing operations. Laurent also mentions, in passing, the polarized, multiplicative-additive fragment MALLP of LLP. This happens to be the main syntax of Girard's theory of Ludics (without weakening). A fundamental point of MALLP, is that the polarity-changing operations $\uparrow$ and $\downarrow$ are more primitive than the full exponentials. On the one hand, O. Laurent shows that Selinger's Control Categories [40] provide an adequate model for LLP. However, the question of natural categorical models for MALLP remained open. In [35], a notion of *linear control category* model for MALLP is introduced; however in order to construct this model, some canonical morphisms which do not live in the syntax must be introduced.

In this paper we introduce what we believe is a natural categorical framework for MALLP. It is based on an ambient $*$-autonomous category $\mathcal{C}$, along with reflective (resp. coreflective) subcategories $\mathcal{C}_-$ and $\mathcal{C}_+$ of polarized objects, together with their associated adjunction and bimodule structure. More generally still, if we do not assume the ambient category $\mathcal{C}$ is $*$-autonomous, we obtain essentially (i.e. with a slight modification) the structures first discussed by M. Barr in his book [7] under the name *pre-$*$-autonomous situations*. These are discussed in Section 3 and Appendix 1 below. Recently, Cockett and Seely [15] have introduced their notion of *polarized categories* motivated by AJ games ([2]), with an elaborate theory of focalized syntax, (two-sided) proof nets, and abstract games based on categorical proof theory. Our work was begun independently of theirs, as an attempt to directly model O. Laurent's MALLP. In conversations with R. Cockett and R. Seely, we now understand our framework to be a special case of their more general one: some connections with their work will be discussed below. Also we have recently become aware that structures similar to ours have arisen in (mostly unpublished) work of Melliès and Selinger [36, 37] again inspired by game semantics. Thus the kinds of structures we deal with here, which essentially go back to M. Barr in a totally different setting, seem to be a natural framework for polarized logics.

The novelties of our paper are in Sections 5 through 7. In Section 5, we present many concrete non-game-theoretic examples of our framework, including Ehrhard's hypercoherences [18] (studied in our previous MALL full completeness work [10]), Chu spaces, and various models based on double gluing and iterated double gluing. In Sections 6,7 we begin our main focus: a study of full completeness theorems (as in our [10]) for polarized logics. By using Game Semantics, O. Laurent has found various full completeness theorems for LLP. However to the best of our knowledge, there are no such full completeness theorems for the fragment MALLP, where polarity shifting operators do not come from exponentials. So this paper is a first step in this program. However the problem turns out to be rather subtle (as we discuss below) so we have chosen to primarily discuss the *multiplicative fragment* MLLP, using a polarized version of functorial polymorphism [6, 11] on our hypercoherence model. This framework is developed in Section 6.

We distinguish between full and weakly full completeness as in Läuchli semantics [26, 11]. The main point of the category **HCoh** of hypercoherences is that, unlike coherence spaces, there are nontrivial natural polarized subcategories $\mathbf{HCoh}_-$ and $\mathbf{HCoh}_+$ with an adjunction between them. The category of coherent spaces **Coh** turns out to be a common subcategory of *both* of these polarized subcategories (in fact, it is a fixed point of the adjunction between $\mathbf{HCoh}_-$ and $\mathbf{HCoh}_+$). Having **Coh** as a common subcategory of the polarized subcategories permits us to reduce the full completeness problem for our polarized hypercoherence model for MLLP to the ordinary multiplicative full completeness problem for **Coh**, which was solved by A. Tan [41]. In Subsection 7.1 we prepare the background by introducing a polarized (MLLP) version of Joyal's softness (with respect to removability of $\uparrow$). In Subsection 7.2 we prove a version of full completeness for $\mathsf{MLLP} + \mathsf{Mix_p}$ and in 7.3 the Main Theorem extends these results to double gluing over hypercoherences, and related structures, and removes the polarized Mix rule. Finally, in Subsection 7.4 we observe the curious result that the polarized hypercoherence model does not kill polarized versions of Gustave functions in MALLP, unlike the nonpolarized case [10], which leads to the failure of polarized full completeness for the full theory MALLP in this model. We end with some open problems on extensions to MALLP.

2 Syntax of Polarized MALL

In polarized (linear) logics, formulas are divided into two classes: *positive* and *negative*. Each of these classes is in turn closed under certain of the logical operations; moreover, there are polarity-changing connectives mapping one class of formulas to the other (and vice versa).

Polarities naturally arise within the proof theory of linear logic. For example, in the case of multiplicative-additive linear logic MALL, we can divide the connectives according to whether their introduction rules are reversible or not [21, 1]. Those connectives which are reversible are called *negative*; those which are not are called *positive*. As we discussed above, positive connectives are the foundation of Andreoli's influential notion of *focalization* in proof search for linear logic [4, 5, 38]. Focalization is a dual property to reversibility.

We now introduce Olivier Laurent's theory of polarized multiplicative-additive linear logic (MALLP). MALLP is a linear fragment (without structural rules) of polarized linear logic (LLP).

Definition 2.1 *Polarized* MALL (MALLP) is defined as follows.

Syntax: *Positive* and *negative* formulas are given by the following BNF notation:

$$\begin{array}{lcl} P & ::= & X \mid P\otimes P \mid P\oplus P \mid \mathbf{1} \mid 0 \mid {\downarrow} N \\ N & ::= & X^{\perp} \mid N \parr N \mid N \& N \mid \perp \mid T \mid {\uparrow} P \end{array}$$

Here $\uparrow$ and $\downarrow$ are called *polarity shifting* operations. Note that $\mathbf{1}$ and 0 are the units of $\otimes$ and $\oplus$, respectively (and dually for $\perp$ and T with respect to $\parr$ and &).

Rules of MALLP are defined as follows: (in the following rules, M and N range over negative formulas and P and Q over positive formulas).

$$\vdash N, N^{\perp} \qquad \frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P\otimes Q}\otimes \qquad \frac{\vdash \Gamma, N, M}{\vdash \Gamma, N \parr M}\parr$$

$$\frac{\vdash \Gamma, N \quad \vdash \Gamma, M}{\vdash \Gamma, N\& M}\& \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, P\oplus Q}\oplus_1 \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P\oplus Q}\oplus_2$$

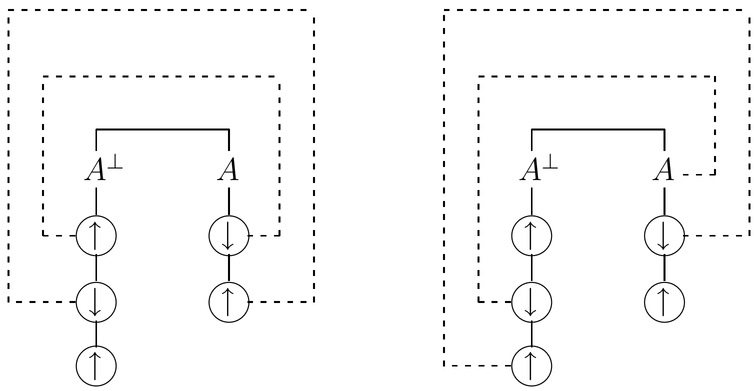
$$\frac{\vdash N, \mathcal{N}}{\vdash {\downarrow} N, \mathcal{N}}\downarrow \qquad \frac{\vdash P, \Gamma}{\vdash {\uparrow} P, \Gamma}\uparrow \qquad \frac{\vdash \Gamma, N \quad \vdash \Delta, N^{\perp}}{\vdash \Gamma, \Delta}\mathrm{cut}$$

where $\mathcal{N}$ consists only of negative formulas.

$$\frac{}{\vdash \Gamma, T}T \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\perp \qquad \frac{}{\vdash \mathbf{1}}\mathbf{1}$$

Example 2.2 Here are two proofs in MALLP, along with their proof net representations. In our general categorical framework to be introduced below, they are interpreted by two different morphisms in general (cf. Example 4.3). We briefly mention models distinguishing these two proofs in Section 8.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^{\perp}, A}{\vdash \uparrow A^{\perp}, A}\uparrow}{\vdash \uparrow A^{\perp}, \downarrow A}\downarrow}{\vdash \uparrow A^{\perp}, \uparrow\downarrow A}\uparrow}{\vdash \downarrow\uparrow A^{\perp}, \uparrow\downarrow A}\downarrow}{\vdash \uparrow\downarrow\uparrow A^{\perp}, \uparrow\downarrow A}\uparrow \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^{\perp}, A}{\vdash \uparrow A^{\perp}, A}\uparrow}{\vdash \downarrow\uparrow A^{\perp}, A}\downarrow}{\vdash \uparrow\downarrow\uparrow A^{\perp}, A}\uparrow}{\vdash \uparrow\downarrow\uparrow A^{\perp}, \downarrow A}\downarrow}{\vdash \uparrow\downarrow\uparrow A^{\perp}, \uparrow\downarrow A}\uparrow$$



The dotted lines in the proof nets denote $\downarrow$-boxes, which correspond to the $\downarrow$-rule in MALLP. These were introduced by O. Laurent [34]. Note that the two proofs above lie in the *multiplicative fragment* of MALLP, which we can define precisely as follows:

Definition 2.3 The theory MLLP (*polarized multiplicative linear logic*) is the subtheory of MALLP in which there are no additive connectives & and $\oplus$.

The following theorem is an important proof-theoretical property of MALLP, proved in [34, 35]:

Proposition 2.4 (Focalization Property) *If* $\vdash \Gamma$ *is provable in* MALLP*, then the sequence* Γ *contains at most one positive formula.*

Syntactic Negation: Following O. Laurent, we adjoin to MALLP a syntactic strictly involutive negation on all formulas by general de Morgan duality. Thus we introduce formal negation, also denoted by $(\ )^{\perp}$, as follows: $X^{\perp\perp} = X$ for atoms X, and we assume $\{\otimes, \parr\}$ and $\{\&, \oplus\}$ are de Morgan duals as in linear logic. Similarly the multiplicative and additive units are dual: $\mathbf{1}^{\perp} = \perp$, $\perp^{\perp} = \mathbf{1}$, $0^{\perp} = T$, $T^{\perp} = 0$. Finally $(\downarrow A)^{\perp} = \uparrow A^{\perp}$ and $(\uparrow A)^{\perp} = \downarrow A^{\perp}$ for any formula A. Positivity and negativity of formulas may be defined as before, after cancelling any occurrences of double-negations.

We now show how MALLP with the above syntactic negation can be given a natural categorical modelling.

3 The Categorical Framework

In this section we present a categorical framework for proofs in polarized MALL, based on a notion of *categorical bimodule*. We call our models *polarized categories*[1]; they are a slightly modified version of M. Barr's *pre-$*$-autonomous situations* [7] (p. 15). A more abstract theory of bimodule models for polarized logics (based on AJ games) is developed in a recent paper of Cockett and Seely [15]. There is also a related analysis of games models in unpublished work of Melliès and Selinger, sketched in [36, 37]. Our models were independently designed for the syntax of O. Laurent's MALLP (for discussion of the literature, see Remark 3.2 and Appendix 1 below.)

Recall, if $\mathcal{C}$ is a $*$-autonomous category with products (i.e. a model of MALL [8, 12]), then tensor and cotensor (par) are functors $\otimes, \invamp : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, along with the dualizing functor $(-)^{\perp} : \mathcal{C}^{op} \to \mathcal{C}$. We will use the same notation for the connectives of the syntax of MALLP and their denotation in our models. For the models we discuss in this paper, we need only consider polarized categories in which the ambient category $\mathcal{C}$ is $*$-autonomous. This simplifies the presentation. We have put our general definition of polarized category, which inspires the treatment below, in Appendix 1. In particular we assume the coherence conditions in Barr's monograph.

Definition 3.1 (Polarized $*$-Autonomous Categories) A *polarized $*$-autonomous category (with products)*, denoted $\mathcal{C}_{+,-}$, consists of the following data:

- A $*$-autonomous category $\mathcal{C}$ with products (and hence coproducts).
- A full subcategory $\mathcal{C}_+$ of $\mathcal{C}$ (called the *positive subcategory*) which is closed under the positive operations $\otimes$ and $\oplus$, along with their respective units $\mathbf{1}$ and 0, along with the induced monoidal structure with respect to both connectives.
- A full subcategory $\mathcal{C}_-$ of $\mathcal{C}$ (called the *negative subcategory*), which is closed under negatives $\invamp$ and &, with their respective units $\perp$ and T, along with the induced monoidal structure with respect to both connectives.
- The contravariant equivalence $(\ )^{\perp}$ on $\mathcal{C}$ induces a contravariant equivalence of the two subcategories:
$$(-)^{\perp} : (\mathcal{C}_+)^{op} \xrightarrow{\cong} \mathcal{C}_-$$
Following Barr [7], p. 15, the equivalence induces a natural isomorphism $A \cong A^{\perp\perp}$ for every positive/negative A.
- The subcategories $\mathcal{C}_-$ (resp. $\mathcal{C}_+$) are reflective (resp. coreflective) subcategories of $\mathcal{C}$. That is, there are distinguished functors
$$\begin{aligned} \uparrow :&\ \ \mathcal{C} \longrightarrow \mathcal{C}_- \\ \downarrow :&\ \ \mathcal{C} \longrightarrow \mathcal{C}_+ \end{aligned}$$

[1]Unfortunately our terminology conflicts with Cockett and Seely's.

satisfying: $\downarrow$ is right adjoint to the inclusion $\mathsf{Inj}_+ : \mathcal{C}_+ \hookrightarrow \mathcal{C}$ and $\uparrow$ is left adjoint to the inclusion $\mathsf{Inj}_- : \mathcal{C}_- \hookrightarrow \mathcal{C}$, i.e.

$$\begin{aligned} \mathcal{C}(\mathsf{Inj}_+(P), X) &\cong \mathcal{C}_+(P, \downarrow X) &(1)\\ \mathcal{C}(X, \mathsf{Inj}_-(N)) &\cong \mathcal{C}_-(\uparrow X, N) &(2)\end{aligned}$$

for all $P \in \mathcal{C}_+$, $X \in \mathcal{C}$ and $N \in \mathcal{C}_-$.

The units and counits of the adjunction (1) are given by:

$$\begin{aligned} \eta^{\downarrow}_P : P \to \downarrow \mathsf{Inj}_+(P), &\text{ also denoted } \eta^{\downarrow}_P : P \to \downarrow P &(3)\\ \varepsilon^{\downarrow}_X : \mathsf{Inj}_+(\downarrow X) \to X, &\text{ also denoted } \varepsilon^{\downarrow}_X : \downarrow X \to X &(4)\end{aligned}$$

and similarly for the adjunction (2); i.e.,

$$\begin{aligned} \eta^{\uparrow}_X : X \to \mathsf{Inj}_-(\uparrow X), &\text{ also denoted } \eta^{\uparrow}_X : X \to \uparrow X &(5)\\ \varepsilon^{\uparrow}_N : \uparrow \mathsf{Inj}_-(N) \to N, &\text{ also denoted } \varepsilon^{\uparrow}_N : \uparrow N \to N &(6)\end{aligned}$$

- De Morgan duality for $\downarrow$ and $\uparrow$

$$\begin{aligned} (\downarrow X)^{\perp} &\cong \uparrow X^{\perp}\\ (\uparrow X)^{\perp} &\cong \downarrow X^{\perp}\end{aligned}$$

$\square$

Let us make some remarks on this definition. First observe that the above adjointnesses (1) and (2) may be combined into the following diagram:

$$\begin{tikzcd} \mathcal{C}_- \arrow[r, hook, bend right, "\mathsf{Inj}_-"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "\uparrow"'] \arrow[r, bend right, "\downarrow"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_+ \arrow[l, hook', bend right, "\mathsf{Inj}_+"'] \end{tikzcd} \qquad (7)$$

We write $\Uparrow$ for $\uparrow \circ \mathsf{Inj}_+$ and $\Downarrow$ for $\downarrow \circ \mathsf{Inj}_-$. Then we may write the above diagram by:

$$\begin{tikzcd} \mathcal{C}_- \arrow[r, bend right, "\Downarrow"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_+ \arrow[l, bend right, "\Uparrow"'] \end{tikzcd}$$

The units and the counits of this adjunction are as follows:

$$\begin{aligned} \eta^{\Uparrow\Downarrow}_P : P \to \Downarrow\Uparrow P &\text{ also denoted } P \to \downarrow\uparrow P &(8)\\ \varepsilon^{\Uparrow\Downarrow}_N : \Uparrow\Downarrow N \to N &\text{ also denoted } \uparrow\downarrow N \to N. &(9)\end{aligned}$$

They are definable by

$$\eta^{\Uparrow\Downarrow}_P = \downarrow(\eta^{\uparrow}_{\mathsf{Inj}_+(P)}) \circ \eta^{\downarrow}_P \quad \text{and dually} \quad \varepsilon^{\Uparrow\Downarrow}_N = \varepsilon^{\downarrow}_N \circ \uparrow(\varepsilon^{\downarrow}_{\mathsf{Inj}_-(N)})$$

Finally, let us remark on strictness (i.e. to what extent the natural isomorphisms in the above definitions can be replaced by equalities). Up to categorical equivalence, we may assume, without loss, that *all $*$-autonomous structure is strict*, in particular that double negation $(\ )^{\perp\perp}$ is *strictly* involutive, rather than up-to-isomorphism (see also Proposition 3.3 below). This causes no problem, by recent coherence theorems for $*$-autonomous categories with units ([14, 28, 16]). Moreover, as pointed out in Proposition 5.35 below, all the models in this paper actually satisfy more: namely that the entire polarized structure is strict.

Remark 3.2 (On Bimodules) Although this work was begun independently of that of R. Cockett and R. Seely, it turns out that our framework is a very special case of their general Polarized Categories [15] (as they remark in their Example 4.4.3). In particular, to compare with their work, note that diagram (7) above determines a *profunctor* (also called a *distributor* or *bimodule*, see [13]) of a particular kind (since the two subcategories $\mathcal{C}_-$ and $\mathcal{C}_+$ are respectively reflective and coreflective subcategories of $\mathcal{C}$).

Profunctors are genuine functors of the form $\phi : \mathcal{C}_+^{op} \times \mathcal{C}_- \to Set$. We think of them as "generalized relations" , denoted[2] $\phi : \mathcal{C}_+ \multimap\!\!\!\rightarrow \mathcal{C}_-$. An instantiation $\phi(P, N)$ is thought of as a set of "formal maps" from P to N, which is closed under left composition (resp. right composition) with genuine maps from $\mathcal{C}_-$ (resp. $\mathcal{C}_+$). If $P \in \mathcal{C}_+$ and $N \in \mathcal{C}_-$, we write $P \multimap\!\!\!\rightarrow N$ for a typical element of $\phi(P, N)$.

As a useful mnemonic, the following patterns (called *legal patterns*) of maps are allowed as MALLP proofs: $P \to N, N \to N, P \to P$ where P and N stand for respectively positive and negative formulas. In our framework, these patterns (in the order given) translate into saying that the usual hom functor $(P, N) \mapsto \mathcal{C}(P, N) \cong \mathcal{C}_-(\uparrow P, N) \cong \mathcal{C}_+(P, \downarrow N)$ is an allowed profunctor. However, the pattern $N \to P$ is not allowed as a MALLP proof pattern (because of the focalization property: see Proposition 2.4). Thus the bimodule $(P, N) \mapsto \mathcal{C}(\uparrow P, \downarrow N)$ cannot be used in our setting.

To keep our discussion general (allowing for more general bimodules), we denote by $\widehat{\mathcal{C}}$ the set of modules on $\mathcal{C}_{+,-}$ in the sense above. We shall be explicit in which bimodule properties we need, allowing for future generalizations. *But for the purposes of this paper, bimodules are given by hom-functors of $\mathcal{C}$, as above.*

The concrete models considered in this paper are polarized $*$-autonomous categories $\mathcal{C}_{+,-}$, which arise from an ambient $*$-autonomous category $\mathcal{C}$. In fact, this structure is somewhat stronger than we actually need; for example, we may remove the assumption that $\mathcal{C}$ is $*$-autonomous. The precise details of these more general models, which we also call *polarized categories*, are in Appendix 1, Section 10.1. There it is pointed out that this framework, is a variation of the original notion of *pre-$*$-autonomous situation* due to M. Barr (see his book [7]), where it is a precursor to his theory of $*$-autonomous categories. This notion suffices for our purposes here, although it is still more specialized than the similarly-named structures in Cockett and Seely [15].

Proposition 3.3 (de Morgan Laws) *From the $*$-autonomous structure of $\mathcal{C}$ with products, we have the following natural isomorphisms:*

$$
\begin{array}{cc}
(P_1 \otimes P_2)^\perp \cong P_1^\perp \parr P_2^\perp & (P_1 \oplus P_2)^\perp \cong P_1^\perp \& P_2^\perp \\
(N_1 \parr N_2)^\perp \cong N_1^\perp \otimes N_2^\perp & (N_1 \& N_2)^\perp \cong N_1^\perp \oplus N_2^\perp
\end{array}
$$
$$
\mathcal{C}_+(P, Q) \cong \mathcal{C}_-(Q^\perp, P^\perp)
$$

[2]We use the opposite notational convention from Borceux [13]

Moreover, the de Morgan laws may be taken as strict equalities, using the coherence result for $*$-autonomous categories $\mathcal{C}$ in Cockett, Hasegawa and Seely [14] (cf. also [28, 16, 17]).

Observe that in the case of bimodules $\widehat{\mathcal{C}}$, the contravariant equivalence $(\ )^{\perp}$ maps legal patterns to legal patterns, in the sense of Remark 3.2 .

Remark 3.4 (On the adjunction $\Uparrow\dashv\Downarrow$) In addition to the work of Cockett and Seely [15], it has recently been pointed out to us that Melliès [36] presented a similar adjunction $\Uparrow\dashv\Downarrow$ to that of (7), but in his case arising from games. He models lifting operators between positive and negative Conway games arising from his categorical formulation of Blass's problem in game semantics. Another similar adjunction also plays a fundamental role for continuation-passing-style models of $\lambda\mu$-calculus (e.g. Selinger's control categories [40] as well as their linear variants by Laurent [35]). However, it appears that this adjunction is a derived property in the setting of O. Laurent, rather than a primitive notion as it is for us.

Let us consider the adjoint equivalence in Lambek-Scott (Proposition 4.2 and Slogan V (p.18) of [33]) when applied to our framework.

Remark 3.5 (Adjoint equivalence in Polarized Categories) An adjunction (F, G, η, ϵ) between categories $\mathcal{A}$ and $\mathcal{B}$ induces an *adjoint equivalence*

$$\mathsf{Fix}\,\eta \quad \cong \quad \mathsf{Fix}\,\epsilon$$

between the fixed point full subcategories $\mathsf{Fix}\,\eta$ and $\mathsf{Fix}\,\epsilon$;

$$\begin{array}{rcl} \mathsf{Fix}\,\eta & = & \{A \in \mathcal{A} \mid \eta(A) \text{ is an iso}\} \\ \mathsf{Fix}\,\epsilon & = & \{B \in \mathcal{B} \mid \epsilon(A) \text{ is an iso}\}, \end{array}$$

where η and ϵ are the unit and counit of the adjunction:

$$\begin{array}{rcl} \eta : id_{\mathcal{A}} & \longrightarrow & GF \\ \epsilon : FG & \longrightarrow & id_{\mathcal{B}} \end{array}$$

Obviously, there is a natural injection between two subcategories of $\mathcal{A}$:

$$\mathsf{Fix}\,\eta \quad \subseteq \quad GF(\mathcal{A}) := \{GF(A) \mid A \in \mathcal{A}\}$$

And similarly for $\mathcal{B}$:

$$\mathsf{Fix}\,\epsilon \quad \subseteq \quad FG(\mathcal{B}) := \{FG(B) \mid B \in \mathcal{A}\}$$

The injection is not surjective in general unless η_{GF} (equivalently ϵ_{FG}) is an isomorphism. In our concrete example of $\mathbf{HCoh}_{+,-}$ to be discussed below, it becomes surjective, which yields a nice characterization of $\mathsf{Fix}\,\eta$; i.e., $\mathsf{Fix}\,\eta$ is equivalent to the category of Girard's coherence spaces, the original semantics of linear logic (see. Proposition 5.15).

Let us sketch the adjoint equivalence applied to a polarized category $\mathcal{C}$ by taking $F :=\Uparrow$ and $G :=\Downarrow$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, phantom, "="] \arrow[d, shift right=2, "\uparrow"'] \arrow[d, phantom, "\dashv"] & \mathcal{C} \arrow[d, shift left=2, "\downarrow"] \arrow[d, phantom, "\dashv"] \\
\mathcal{C}_- \arrow[u, hook, shift right=2, "\mathsf{Inj}_-"'] \arrow[r, shift right=2, "\Downarrow"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_+ \arrow[u, hook, shift left=2, "\mathsf{Inj}_+"] \arrow[l, shift right=2, "\Uparrow"'] \\
\uparrow\downarrow\mathcal{C} \arrow[u, hook] & \downarrow\uparrow\mathcal{C} \arrow[u, hook] \\
\mathsf{Fix}\,\epsilon \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathsf{Fix}\,\eta \arrow[u, hook]
\end{tikzcd}$$

In general, it is not the case that $\mathsf{Fix}\,\eta$ (resp. $\mathsf{Fix}\,\epsilon$) coincides with $\mathcal{C}_+$ (resp. $\mathcal{C}_-$) (cf. Remark 5.14 in Section 5.1). This is because in the syntax of MALLP, the sequent $\downarrow\uparrow P \vdash P$ is not provable. Moreover we will observe in the following Section 8 that in $\mathcal{C}_+$ (resp. in $\mathcal{C}_-$) there exists a decreasing sequence $\{\ (\downarrow\uparrow)^n\mathcal{C}\ \}_{n\geq 1}$ (resp. $\{\ (\uparrow\downarrow)^n\mathcal{C}\ \}_{n\geq 1}$) of subcategories containing $\mathsf{Fix}\,\eta$ (resp. $\mathsf{Fix}\,\epsilon$) (cf. Definition 8.1).

4 Interpretations of Proofs and Soundness

We interpret proofs of MALLP in a polarized $*$-autonomous category $\mathcal{C}_{+,-}$ as follows:

- Negative (resp. positive) formulas are interpreted as objects in $\mathcal{C}_-$ (resp. $\mathcal{C}_+$) in the obvious way.
- A sequent $\vdash \Gamma$ in MALLP will be interpreted as some homset $[\![\vdash \Gamma]\!]$ in $\mathcal{C}$. Given an interpretation $[\![-]\!]$, a *proof* of a sequent $\vdash \Gamma$ is an element in $[\![\vdash \Gamma]\!]$ and a sequent $\vdash \Gamma$ is *provable* if it has a proof (i.e. if the set $[\![\vdash \Gamma]\!]$ is non-empty.)
- We will interpret the proofs of provable sequents in MALLP as either (elements of) bimodules $\widehat{\mathcal{C}}$ or as morphisms of $\mathcal{C}_-$ (or dually $\mathcal{C}_+$) as follows:
  - $[\![\vdash N_1, \ldots, N_k]\!] : \ \mathbf{1} \multimap [\![N_1]\!] \parr [\![N_2]\!] \parr \cdots \parr [\![N_k]\!]$ in $\widehat{\mathcal{C}}$
  - $[\![\vdash N_1, \ldots, N_k, P]\!] : \ [\![P]\!]^\perp \longrightarrow [\![N_1]\!] \parr [\![N_2]\!] \parr \cdots \parr [\![N_k]\!] \in \mathcal{C}_-$.

Remark 4.1 (Maps vs. Bimodules) A property of our framework is that there are various formal connections between maps and bimodules. Similar observations are made by Cockett and Seely in their setting (see [15], Sections 2 and 3).

1. Notice that by monoidal closedness of $\mathcal{C}$, for bimodules induced by the hom functor, we have the following *bijection of bimodules* in $\widehat{\mathcal{C}}$:

$$\frac{\widehat{\mathcal{C}}(\mathbf{1}, N_1 \parr \cdots \parr N_k)}{\overline{\widehat{\mathcal{C}}(N_1^\perp \otimes \cdots \otimes N_l^\perp, N_{l+1} \parr \cdots \parr N_k)}}$$

   This will be used in interpreting MALLP sequents of the form $\vdash N_1, \ldots, N_k$.

2. By the duality between $\mathcal{C}_+$ and $\mathcal{C}_-$, we have the following bijection of homsets:

$$\frac{\mathcal{C}_-(P^\perp, N_1 \parr \cdots \parr N_k)}{\overline{\mathcal{C}_+(N_1^\perp \otimes \cdots \otimes N_k^\perp, P)}}$$

3. Finally, our framework $\mathcal{C}_{+,-}$ supports *bijections* between modules and maps, as follows:

$$\frac{\dfrac{\mathcal{C}_+(P, \downarrow N)}{\overline{\widehat{\mathcal{C}}(P, N)}}}{\overline{\mathcal{C}_-(\uparrow P, N)}}$$

   In particular, since our modules are given by $\mathcal{C}$-homsets, these bijections are given by the adjunctions $\mathsf{Inj}_+ \dashv \downarrow$ and $\uparrow \dashv \mathsf{Inj}_-$. Thus we can apply the functors $\uparrow$ and $\downarrow$ to maps in modules.

We now interpret formal MALLP proofs as follows:

1. *Axiom*: $[\![\vdash N^\perp, N]\!] = id_{[\![N]\!]} : [\![N^\perp]\!]^\perp \to [\![N]\!]$ in $\mathcal{C}_-$ (up to isomorphism), since $[\![N^\perp]\!] \in \mathcal{C}_+$ hence $[\![N^\perp]\!]^\perp \cong [\![N]\!]$ in the $*$-autonomous category $\mathcal{C}$, hence in $\mathcal{C}_-$.

2. *Linear connectives:* $\otimes, \parr, \oplus, \&$-rules are interpreted (using duality) from the induced monoidal structure on $\mathcal{C}_+$ and $\mathcal{C}_-$.

3. *Polarity Changing* $\downarrow$*:* We use the monoidal closure, and adjunction structure $\uparrow \dashv \mathsf{Inj}_-$ of $\mathcal{C}_{+,-}$:

$$\begin{array}{ccll} \mathbf{1} & \multimap\!\!\to & [\![N]\!] \parr (\parr \mathcal{N}) & \text{in } \widehat{\mathcal{C}} \\ \hline\hline [\![N]\!]^\perp & \multimap\!\!\to & \parr \mathcal{N} & \text{in } \widehat{\mathcal{C}} \\ \hline\hline \uparrow [\![N]\!]^\perp & \longrightarrow & \parr \mathcal{N} & \text{in } \mathcal{C}_- \end{array}$$

   where $[\![\downarrow N]\!]^\perp \cong \uparrow [\![N]\!]^\perp$. Double lines refer to reversible inferences.

4. *Polarity Changing* $\uparrow$*:*

$$\frac{\dfrac{\downarrow [\![P]\!]^\perp \multimap\!\!\to [\![P]\!]^\perp \quad \text{in } \widehat{\mathcal{C}} \qquad [\![P]\!]^\perp \longrightarrow \parr [\![\Gamma]\!] \quad \text{in } \mathcal{C}_-}{\downarrow [\![P]\!]^\perp \multimap\!\!\to \parr [\![\Gamma]\!] \qquad \text{in } \widehat{\mathcal{C}}}}{\overline{\mathbf{1} \multimap\!\!\to \uparrow [\![P]\!] \parr (\parr [\![\Gamma]\!]) \qquad \text{in } \widehat{\mathcal{C}}}}$$

   where $(\downarrow [\![P]\!]^\perp)^\perp \cong \uparrow [\![P]\!]$ in $\mathcal{C}_-$. Notice the first line is closure (under right multiplication) of the bimodule corresponding to the counit (4) of the adjunction $\mathsf{Inj}_+ \dashv \downarrow$ in $\mathcal{C}$ with an arrow in $\mathcal{C}_-$, so the inference is not reversible. In our concrete modules given by homfunctors, this is simply composition in $\mathcal{C}$.

Proposition 4.2 (Soundness)

1. *If* $\vdash \Gamma$ *is provable in* MALLP *then* $[\![\vdash \Gamma]\!]$ *is nonempty.*

2. $[\![-]\!]$ *is an invariant of cut-elimination, i.e. if* $\Pi \rightsquigarrow \Pi'$ *is a cut-elimination reduction in* MALLP*, then* $[\![\Pi]\!] = [\![\Pi']\!]$.

Proof. (1) is obvious from the inductive interpretation of proofs. As for (2), we illustrate one of the crucial steps. Consider the following cut-elimination step:

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{\vdash N, \mathcal{N}}}{\vdash\downarrow N, \mathcal{N}}\downarrow \quad \dfrac{\dfrac{\vdots\ \pi_2}{\vdash N^\perp, \mathcal{M}}}{\vdash\uparrow N^\perp, \mathcal{M}}\uparrow}{\vdash \mathcal{M}, \mathcal{N}}\,cut \qquad \rhd \qquad \dfrac{\dfrac{\vdots\ \pi_1}{\vdash N, \mathcal{N}} \quad \dfrac{\vdots\ \pi_2}{\vdash N^\perp, \mathcal{M}}}{\vdash \mathcal{N}, \mathcal{M}}\,cut$$

We interpret the left branch of the left cut as follows (for typographical reasons, we write $\mathcal{N}$ rather than $\parr\ \mathcal{N}$ and $\uparrow \mathcal{N}$ for $\uparrow (\parr\ \mathcal{N})$:

$$\dfrac{\dfrac{[\![\pi_1]\!] : N^\perp \multimap\!\!\!\!\rightarrow \mathcal{N} \in \widehat{\mathcal{C}}}{\uparrow([\![\pi_1]\!]) : \ \uparrow N^\perp \multimap\!\!\!\!\rightarrow\ \uparrow \mathcal{N} \in \widehat{\mathcal{C}}\,,} \qquad \varepsilon^\uparrow_{\mathcal{N}} :\uparrow \mathcal{N} \to \mathcal{N} \in \mathcal{C}_-}{\varepsilon^\uparrow_{\mathcal{N}} \circ \uparrow([\![\pi_1]\!]) :\uparrow N^\perp \multimap\!\!\!\!\rightarrow \mathcal{N} \in \widehat{\mathcal{C}}}$$

Similarly, we will interpret the right branch of the left cut to obtain $[\![\pi_2]\!] \circ \varepsilon^\downarrow_N :\downarrow N \multimap\!\!\!\!\rightarrow \mathcal{M} \in \widehat{\mathcal{C}}$. Thus we will interpret the entire proof on the LHS of $\rhd$ as the following bimodule element $([\![\pi_2]\!] \circ \varepsilon^\downarrow_N) \circ (\varepsilon^\uparrow_{\mathcal{N}} \circ \uparrow([\![\pi_1]\!]))^\perp : \mathcal{N}^\perp \multimap\!\!\!\!\rightarrow \mathcal{M} \in \widehat{\mathcal{C}}$. We then obtain:

$$\begin{array}{rcl}
([\![\pi_2]\!] \circ \varepsilon^\downarrow_N) \circ (\varepsilon^\uparrow_{\mathcal{N}} \circ \uparrow([\![\pi_1]\!]))^\perp & = & ([\![\pi_2]\!] \circ \varepsilon^\downarrow_N) \circ (\uparrow [\![\pi_1]\!])^\perp \circ (\varepsilon^\uparrow_{\mathcal{N}})^\perp \\
& = & ([\![\pi_2]\!] \circ \varepsilon^\downarrow_N) \circ (\uparrow [\![\pi_1]\!])^\perp \circ \eta^\downarrow_{\mathcal{N}^\perp} \\
& = & [\![\pi_2]\!] \circ (\varepsilon^\downarrow_N \circ (\uparrow [\![\pi_1]\!])^\perp \circ \eta^\downarrow_{\mathcal{N}^\perp}) \\
& \cong & [\![\pi_2]\!] \circ [\![\pi_1]\!]^\perp.
\end{array}$$

The last isomorphism is based on the fact that $(\varepsilon^\downarrow_N \circ (\uparrow [\![\pi_1]\!])^\perp \circ \eta^\downarrow_{\mathcal{N}^\perp})^\perp = [\![\pi_1]\!]$, up to isomorphism. This follows from the following diagram, which commutes up to isomorphism, using the fact that $\mathcal{C}_-$ is a *reflective* subcategory of $\mathcal{C}$, with reflector $\uparrow$. For ease of reading, we write $G : \mathcal{C}_- \hookrightarrow \mathcal{C}$ for the inclusion Inj_- and $F : \mathcal{C} \to \mathcal{C}_-$ for its left adjoint $\uparrow$. Here η and ε are the canonical adjunctions:

$$\begin{tikzcd}
GFC \arrow[r, "GF{[\![\pi_1]\!]}"] & GFN \arrow[r, "\cong"] & FN \arrow[r, "\cong"] & FGN \arrow[d, "\varepsilon_N"] \\
C \arrow[u, "\eta_C"] \arrow[r, "{[\![\pi_1]\!]}"] & N \arrow[u, "\eta_N"] & GN \arrow[l, "\cong"'] & N \arrow[l, "\cong"']
\end{tikzcd}$$

$\square$

Example 4.3 In $\mathcal{C}_{+,-}$, the two proofs of Example 2.2 are interpreted respectively as follows where $\epsilon_X := \epsilon^{\downarrow}_X$ and $\eta_X := \eta^{\uparrow}_X$:

- left proof

$$\epsilon_{\uparrow\downarrow A^\circ} \uparrow (\eta_{\downarrow A^\circ} \downarrow (id_{A^\circ}\epsilon_A)\circ\eta_{\downarrow A})\circ\epsilon_{\uparrow\downarrow A}$$

- right proof

$$\eta_{\downarrow A^\circ} \downarrow (\epsilon_{A^\circ} \uparrow (id_{A^\circ}\epsilon_A)\circ\epsilon_{\uparrow\downarrow A})\circ\eta_{\downarrow\uparrow\downarrow A}$$

These two morphisms are not equal in general.

Remark 4.4 (Nonfaithful Models) If the adjunction $\uparrow\dashv \mathsf{Inj}_-$ induces an *equality* of hom-sets, i.e. $\mathcal{C}_-(\uparrow P, N) = \widehat{\mathcal{C}}(P, N)$, the model $\mathcal{C}_{+,-}$ turns out to be *nonfaithful.* For example, in the concrete models $\mathbf{HCoh}_{+,-}$ and $(\mathbf{G}^2\mathcal{C})_{+,-}$ discussed below in Section 5, $\uparrow$ acts nontrivally on objects, but is the identity functor on morphisms; such models turn out to be nonfaithful, i.e. such models will identify the two MLLP proofs in Example 2.2. This is in sharp contrast to Olivier Laurent's LLP (see [35]), which is polarized linear logic with exponentials. In this setting, it is well-known that if a categorical semantics identifies the two MLLP proofs (with $\downarrow=!$ and $\uparrow=?$) then the semantics collapses to a poset, i.e. a boolean algebra. Further remarks on the issue of faithfulness are given in Section 8.

In the following section, we shall discuss a wide range of examples of the above categorical modeling of MALLP. Namely, in subsection 5.1 we discuss hypercoherences, in subsection 5.2 we discuss Chu spaces, in subsections 5.3 we discuss double gluing over an arbitrary polarized category $\mathcal{C}_{+,-}$ and in subsection 5.4, iterated double gluing over an arbitrary $*$-autonomous category.

5 Examples of Polarized Categories

In this section we present four examples of our polarized categories. We first introduce some general set-theoretical notation.

Notation 5.1 We write $P(A)$ for the powerset of the set A. We denote the finite power set $P_{fin}(A) := \{\alpha \in P(A) \mid \alpha \text{ is a finite set}\}$. $P^*(A) := P(A) \setminus \{\emptyset\}$ and similarly $P^*_{fin}(A) := P_{fin}(A) \setminus \{\emptyset\}$. We write $X \subseteq^*_{fin} Y$ when X is a finite non-empty subset of Y. Similarly,$X \subseteq^* Y$ means X is a nonempty subset of Y. $A \times B$ denotes the cartesian product of sets A and B. For $C \subseteq A \times B$, we use $\pi_1(C) := \{a \in A \mid \exists b \in B\ (a, b) \in C\}$ for its first projection and $\pi_2(C) := \{b \in B \mid \exists a \in A\ (a, b) \in C\}$ for its second projection. $A + B$ denotes the disjoint union of sets A and B, i.e., $A + B := \{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$. For $C \subseteq A + B$, we denote its two components $C_1 := \{a \in A \mid (1, a) \in D\}$ and $C_2 := \{b \in B \mid (2, b) \in D\}$. Finally, we write $\#A$ for the cardinality of the set A.

5.1 Hypercoherences and Polarities

In this subsection we present a concrete example of a polarized category arising from Ehrhard's $*$-autonomous category of hypercoherences [18]. We begin by recalling the definition of hypercoherence. We follow the treatment in [10, 3, 18]. We then introduce the polarized subcategories of Ehrhard (see Section 5 of [18]).

5.1.1 Hypercoherences

In [10] we introduced the hierarchy of categories of hypercoherences $\mathbf{Coh_n}$ for $2 < n \leq \omega$, which are intermediate between Girard's **Coh** [19], which is $\mathbf{Coh_3}$, and Ehrhard's hypercoherences **HCoh** [18], which is $\mathbf{Coh}_\omega$ in our terminology. For the purposes of this paper, we will primarily consider hypercoherences.

Definition 5.2 (Hypercoherence E) A *hypercoherence* E is a pair

$$E := (|E|, \Gamma(E))$$

where $|E|$ is a set and $\Gamma(E) \subseteq P^*(|E|)$ such that $\forall a \in |E| \quad \{a\} \in \Gamma(E)$.

We use the notation $\Gamma^*(E) := \{u \in \Gamma(E) \mid \#u > 1\}$. A hypercoherence E is identified with a hypergraph, $|E|$ determines the set of nodes and each element of $\Gamma(E)$ determines a hyperedge on $|E|$.

Definition 5.3 (the set $D(E)$ of states for a hypercoherence E) For a hypercoherence E, the set $D(E)$ of *states* for E is

$$D(E) := \{X \subseteq |E| \mid \forall u \subseteq^*_{fin} X \; u \in \Gamma(E)\}$$

where $B \subseteq^* A$ means B is a nonempty subset of A.

Definition 5.4 (linear implication of hypercoherences) For hypercoherences E and F, the hypercoherence $E \multimap F$, called *linear implication* of E and F, is

$$E \multimap F := (|E| \times |F|, \Gamma(E \multimap F))$$

where $w \in \Gamma(E \multimap F)$ iff

(i) $w \subseteq |E| \times |F|$,
(ii) $\pi_1(w) \in \Gamma(E) \Rightarrow (\pi_2(w) \in \Gamma(F) \wedge \; (\#\pi_2(w) = 1 \Rightarrow \#\pi_1(w) = 1))$

Definition 5.5 (HCoh) The category **HCoh** consists of the following:
objects: hypercoherences $E := (|E|, \Gamma(E))$
morphisms: $\mathbf{HCoh}(E, F) := D(E \multimap F)$

Remark 5.6 A morphism is a relation on hypergraphs which sends hyperedges to hyperedges and such that the preimage of a loop is a loop (but in general the preimage of a hyperedge is not necessarily a hyperedge).

For $E, F \in \mathbf{HCoh}$

1. $Id_E := \{(a, a) \mid a \in |E|\} \in D(E \multimap E)$
2. If $R \in D(E \multimap F)$ and $S \in D(F \multimap G)$ then the relational composition

$$S \circ R := \{(a, c) \mid \exists b((a, b) \in R \wedge \; (b, c) \in S)\} \in D(E \multimap G).$$

Proposition 5.7 HCoh *is a $*$-autonomous category with products and coproducts.*

We indicate the structure on objects, following [3]:

(linear negation:) $E^\perp := (|E|, \Gamma(E^\perp))$ where

$$\Gamma^*(E^\perp) := P^*_{fin}(|E|) \setminus \Gamma^*(E).$$

(tensor:) $E \otimes F := (|E| \times |F|, \Gamma(E \otimes F))$ where

$$w \in \Gamma(E \otimes F) \text{ iff } \begin{array}{l} w \subseteq |E| \times |F|, w \text{ is finite and} \\ (w_1 \in \Gamma(E) \wedge \ w_2 \in \Gamma(F)). \end{array}$$

(product:) $E \& F = (|E| + |F|, \Gamma(E \& F))$ where

$$w \in \Gamma(E \& F) \text{ iff } \begin{array}{l} w \subseteq |E| + |F|, w \text{ is finite and} \\ (w_2 = \emptyset \Rightarrow w_1 \in \Gamma(E)) \wedge \ (w_1 = \emptyset \Rightarrow w_2 \in \Gamma(F)). \end{array}$$

Hence we have by de Morgan duality:

(par:) $E \parr F := (|E| \times |F|, \Gamma(E \parr F))$ where

$$w \in \Gamma^*(E \parr F) \text{ iff } \begin{array}{l} w \subseteq |E| \times |F|, w \text{ is finite and} \\ (w_1 \in \Gamma^*(E) \vee w_2 \in \Gamma^*(F)). \end{array}$$

(coproduct:) $E \oplus F := (|E| + |F|, \Gamma(E \oplus F))$ where

$$w \in \Gamma(E \oplus F) \text{ iff } \begin{array}{l} w \subseteq |E| + |F|, w \text{ is finite and} \\ (w_1 \in \Gamma(E) \wedge \ w_2 = \emptyset) \vee (w_1 = \emptyset \wedge \ w_2 \in \Gamma(F)). \end{array}$$

$\mathbf{1}$ denotes the unique hypercoherence such that $|\mathbf{1}|$ is the singleton $\{\star\}$. Then $\mathbf{1} = \mathbf{1}^\perp$ and $\mathbf{1}$ becomes the unit both for $\otimes$ and $\parr$.

5.1.2 The Polarized Category $\mathbf{HCoh}_{+,-}$

We introduce polarized subcategories arising from notions in Ehrhard [18], Section 5.

Definition 5.8 (positive and negative subcategories $\mathbf{HCoh}_+$ and $\mathbf{HCoh}_-$)

- $\mathbf{HCoh}_+$ is the subcategory of $\mathbf{HCoh}$ consisting of *hereditary* hypercoherences. A hypercoherence E is called *hereditary* when the following holds:

$$\forall u \in \Gamma(E)\ \forall v \subseteq^*_{fin} u \ \ v \in \Gamma(E)$$

- $\mathbf{HCoh}_-$ is the subcategory of $\mathbf{HCoh}$ consisting of *antihereditary* hypercoherences. A hypercoherence E is called *antihereditary* when $E^\perp$ is hereditary; i.e., the following holds:

$$\forall u \in \Gamma^*(E) \ \text{ if } \ v \subseteq^*_{fin} |E| \ \text{ is such that } \ u \subseteq v \text{ then } \ v \in \Gamma^*(E)$$

Proposition 5.9

- $\mathbf{HCoh}_+$ *is closed under positives $\otimes$ and $\oplus$; i.e., if E and F are hereditary then so are $E \otimes F$ and $E \oplus F$.*

- **HCoh**$_-$ *is closed under negatives* $⅋$ *and* $\&$; *i.e., if* E *and* F *are antihereditary then so are* $E ⅋ F$ *and* $E\&F$.

We can use the same construction above on n-coherences $\mathbf{Coh_n}$ (see [10]) to obtain $(\mathbf{Coh_n})_+$ and $(\mathbf{Coh_n})_-$, with $3 \leq n \leq \omega$. In the case $n = 3$, $\mathbf{Coh}_3$ is simply **Coh**, the category of coherence spaces. It can be shown that

- $\mathbf{Coh}_+ = \mathbf{Coh}_- = \mathbf{Coh}$,
- For $n > 3$, $(\mathbf{Coh_n})_+ \neq (\mathbf{Coh_n})_-$.

Hence the polarization of $\mathbf{Coh_n}$ only begins at levels beyond 3.

Definition 5.10 (functors $\downarrow$ and $\uparrow$)

- A functor $\downarrow$: $\mathbf{HCoh} \longrightarrow \mathbf{HCoh}_+$ is defined by

  (on objects) For a hypercoherence E,

  $$\Gamma(\downarrow E) := \{u \in \Gamma(E) \mid \forall v \subseteq^*_{fin} u \;\; v \in \Gamma(E)\},$$

  which is a *restriction* of $\Gamma(E)$ in that $\Gamma(\downarrow E) \subseteq \Gamma(E)$.

  (on morphisms) $\downarrow$ is the identity; i.e., for $R \in D(E \multimap F)$, we define $\downarrow R \in D(\downarrow E \multimap \downarrow F)$ by

  $$\downarrow R = R$$

- Dually, a functor $\uparrow$: $\mathbf{HCoh} \longrightarrow \mathbf{HCoh}_-$ is defined by

  (on objects) For a hypercoherence E,

  $$\Gamma^*(\uparrow E) := \{u \subseteq^*_{fin} |E| \mid \exists v \subseteq u \;\; v \in \Gamma^*(E)\},$$

  which is an *expansion* of $\Gamma(E)$ in that $\Gamma(\uparrow E) \supseteq \Gamma(E)$.

  (on morphisms) $\uparrow$ is the identity; i.e., for $R \in D(E \multimap F)$, we define $\uparrow R \in D(\uparrow E \multimap \uparrow F)$ by

  $$\uparrow R = R$$

Proposition 5.11 (An adjunction $\mathsf{Inj}_+ \dashv \downarrow$) *For every object* $E \in \mathbf{HCoh}_+$ *and* $F \in \mathbf{HCoh}$,

$$\mathbf{HCoh}(E, F) = \mathbf{HCoh}_+(E, \downarrow F)$$

That is, the functor $\downarrow$ *is right adjoint to the inclusion functor* $\mathsf{Inj}_+ : \mathbf{HCoh}_+ \longrightarrow \mathbf{HCoh}$.

Dually, we have

Proposition 5.12 (An adjunction $\uparrow \dashv \mathsf{Inj}_-$) *For every object* $E \in \mathbf{HCoh}$ *and* $F \in \mathbf{HCoh}_-$,

$$\mathbf{HCoh}(E, F) = \mathbf{HCoh}_-(\uparrow E, F)$$

That is, the functor $\uparrow$ *is left adjoint to the inclusion functor* $\mathsf{Inj}_- : \mathbf{HCoh}_- \longrightarrow \mathbf{HCoh}$.

From the above two propositions we have:

Corollary 5.13 $\mathbf{HCoh}_{+,-}$ *is a polarized category.*

For every hypercoherence $E \in \mathbf{HCoh}_+$, the unit $\eta : E \longrightarrow \downarrow\uparrow E$ of (8) for the adjunction $\Uparrow \dashv \Downarrow$ yields a natural embedding

$$\Gamma(E) \quad \subset \quad \Gamma(\downarrow\uparrow E) \tag{10}$$

However the converse does not hold in general, (see also Remark 3.5 above), as follows:

Remark 5.14 ($\mathbf{HCoh}_+$ does not coincide with $\mathsf{Fix}\,\eta$) The natural embedding (10) is strict: i.e., there exists a hypercoherence $E \in \mathbf{HCoh}_+$ such that $\quad \Gamma(\uparrow\downarrow E) \not\subset \Gamma(E)$.

Proof. Define $\Gamma^*(E) := \{\{a, b\}, \{b, c\}, \{c, a\}\}$ with $\mid E \mid := \{a, b, c\}$. This is equivalent to $\{\{a, b, c\}\} = \Gamma^*(E^\perp)$. The definition yields that $\{a, b, c\} \in \Gamma^*(\uparrow\downarrow E)$, and is not an element of $\Gamma^*(E)$, as required. $\square$

On the other hand, in the framework of $\mathbf{HCoh}_{+,-}$, we have a nice characterization of the subcategories of $\mathbf{HCoh}_+$ and $\mathbf{HCoh}_-$ sketched in Remark 3.5:

Proposition 5.15 *The following hold for the subcategory* $\mathsf{Fix}\,\eta$ *and* $\downarrow\uparrow\mathbf{HCoh}$*:*

1. $\mathsf{Fix}\,\eta$ *(resp.* $\mathsf{Fix}\,\epsilon$*) coincides with* $\downarrow\uparrow\mathbf{HCoh}$ *(resp.* $\uparrow\downarrow\mathbf{HCoh}$*).*

2. $\mathsf{Fix}\,\eta$ *is equivalent to the category* $\mathbf{Coh}$ *of coherent spaces.*

3. *The category* $\downarrow\uparrow\mathbf{HCoh}$ *is equivalent to the category* $\uparrow\downarrow\mathbf{HCoh}$.

Proof. 1. It suffices to show that η_{GF} is an isomorphism: i.e.,

$$\Gamma^*(\downarrow\uparrow E) \quad = \quad \Gamma^*(\downarrow\uparrow\downarrow\uparrow E)$$

This holds since u belonging to the L.H.S. and the R.H.S is characterized by the following same condition:

$$\forall u' \subseteq u\,(\#u' = 2 \Rightarrow u' \in \Gamma(E)) \tag{11}$$

2. $u \in \Gamma^*(\downarrow\uparrow E)$ if and only if (11) holds. So sending each hypercoherence $\downarrow\uparrow E$ to the coherence consisting of edges from $\Gamma(\downarrow\uparrow E)$ yields the isomorphism between the two categories.

3. From 1 and the adjoint equivalence $\mathsf{Fix}\,\eta \cong \mathsf{Fix}\,\epsilon$.

$\square$

We end this subsection with the following proposition, which will be used for showing the existence of a polarized Gustave function, discussed in Subsection 7.4.

Proposition 5.16 (↓ and ↑ are strict monoidal in HCoh) *In* **HCoh**, *↓ and ↑ induce strict monoidal functors, i.e. we have for hypercoherences E and F,*

$$\begin{aligned}\downarrow(E_1 \otimes E_2) &= \downarrow E_1 \otimes \downarrow E_2 \\ \uparrow(E_1 \parr E_2) &= \uparrow E_1 \parr \uparrow E_2\end{aligned}$$

Proof. Since one is dual to the other, we prove the preservation of $\otimes$:
($\supseteq$) Direct, since $\forall v \subseteq^* u$, we have $v_i \subseteq^* u_i$ with $i \in \{1, 2\}$.
($\subseteq$) For $u \in$ L.H.S, we shall show $u_i \in \Gamma(\downarrow E_i)$; this is derived from the following: $\forall u' \subseteq^* u_i$ $\exists v \subseteq u$ such that $v_i = u'$. □

5.2 Chu Spaces and Polarities

Chu spaces were introduced by M. Barr (and studied by his student P. Chu) in [7] as a formal construction for building $*$-autonomous categories from (finitely complete) symmetric monoidal closed ones. Chu spaces have turned out to be extremely fruitful for building models of linear logic, as well as in applications to theoretical computer science and in mathematical studies of duality theories. For detailed discussions and history, see [7, 8, 39].

There are many categories of Chu spaces, depending on the underlying symmetric monoidal closed category. We now briefly describe the category **Chu**(**Set**, K), a particularly simple one. Let K be a set.

Definition 5.17 A *Chu Space* $\mathcal{A} = (A, R, X)$ consists of a pair of sets A, X and a function $A \times X \xrightarrow{R} K$. Think of R as a "K-valued matrix" (or "K-valued relation") from A to X.

Let $\mathcal{A} = (A, R, X)$ and $\mathcal{B} = (B, S, Y)$ be Chu spaces. A *morphism* of Chu spaces $(f, g) : \mathcal{A} \to \mathcal{B}$ is a pair of maps (f, g), where $f : A \to B$ and $g : Y \to X$, satisfying

$$S(f(a), y) = R(a, g(y)) \qquad \text{for all } a \in A, y \in Y.$$

Chu spaces with morphisms between them form a category, **Chu**(**Set**, K), with composition and identities given pointwise.

Chu(**Set**, K) is a self-dual, complete (thus cocomplete) $*$-autonomous category, with small (co)limits inherited from **Set** (for details, see [8, 39]). For our purposes, we only need the following properties:

Proposition 5.18 **Chu**(**Set**, K) *is a $*$-autonomous category with products, thus a model of* MALL.

Proof. Let us sketch the relevant structure. Let $\mathcal{A} = (A, R, X)$ and $\mathcal{B} = (B, S, Y)$ be Chu spaces.

Linear Negation: $\mathcal{A}^\perp = (X, R^{op}, A)$, where $R^{op} = X \times A \xrightarrow{\cong} A \times X \xrightarrow{R} K$ is given by $R^{op}(x, a) = R(a, x)$. Given $(f, g) : \mathcal{A} \longrightarrow \mathcal{B}$, define $(f, g)^\perp : \mathcal{B}^\perp \longrightarrow \mathcal{A}^\perp$ by $(f, g)^\perp = (g, f)$.

Tensor: $\mathcal{A} \otimes \mathcal{B} = (A \times B, T, hom(\mathcal{A}, \mathcal{B}^\perp))$, where $T(\langle a, b\rangle, (h, k)) = R(a, k(b))(= S(b, h(a))$).

Given $(f,g) : \mathcal{A} \longrightarrow \mathcal{A}'$ and $(u,v) : \mathcal{B} \longrightarrow \mathcal{B}'$, define $(f,g) \otimes (u,v) : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A}' \otimes \mathcal{B}'$ by:

$$(f,g) \otimes (u,v) = (f \times u, (v\circ - \circ f, g\circ - \circ u)\ )$$

where, for example, if $h' : A' \to Y'$, $(v\circ - \circ f)(h') = v\circ h' \circ f : A \to Y$.

Coproducts: $\mathcal{A} \oplus \mathcal{B} = (A + B, F, X \times Y)$, where $F((1,a),(x,y)) = R(a,x)$ and $F((2,b),(x,y)) = S(b,y)$. The unit for $\oplus$ (i.e. the initial object in $\mathbf{Chu}(\mathbf{Set}, K)$) is $0 = (\emptyset, !, \{*\})$. Injections $\mathtt{in}_\ell : \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathcal{B}$ (and similarly right injections) and copairing $[(f,g),(u,v)] : \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{C}$ are easy.

Finally, products are given by de Morgan duality from coproducts. $\square$

Given a Chu space $\mathcal{A} = (A, R, X)$, let $\widehat{R} : A \to K^X$ denote the currying of R, so that $\widehat{R}(a)(x) = R(a,x)$. Call $\widehat{R}(a)$ the "row determined by a" (in the K-valued matrix of R). Dually, define $\check{R} : X \to K^A$ to be the currying of R^{op}, so $\check{R}(x)(a) = R(a,x)$. We call $\check{R}(x)$ the "column determined by x".

Definition 5.19 A Chu space $\mathcal{A}$ is *separated* if $\widehat{R}$ is injective. Dually, a Chu space is *extensional* if $\check{R}$ is injective. Let $\mathbf{Chu}_{sep}$ be the full subcategory of $\mathbf{Chu}(\mathbf{Set}, K)$ of separated Chu spaces. Similarly, let $\mathbf{Chu}_{ex}$ be the full subcategory of $\mathbf{Chu}(\mathbf{Set}, K)$ of extensional Chu spaces.

Note: from a matrix viewpoint, extensional Chu spaces have no repeated columns in R, while separated Chu spaces have no repeated rows.

Definition 5.20 (Separated and Extensional Collapses) Let $\mathcal{A} = (A, R, X)$ be a Chu space. Let $Ker(\widehat{R}) = \{(a,a') \in A \times A \mid \widehat{R}(a) = \widehat{R}(a')\ \}$. This is an equivalence relation on A, denoted by $\sim$, with canonical quotient map $\nu : A \to A/\!\sim$. Define the quotient Chu space $\mathcal{A}/\!\sim\ = (A/\!\sim, R/\!\sim, X)$ where, for $[a] \in A/\!\sim, x \in X$,

$$R/\!\sim\ ([a], x) = R(a,x)$$

One easily checks that $R/\!\sim$ is well defined, that $\mathcal{A}/\!\sim$ is separated, and that there is a canonical quotient morphism $(\nu, id_X) : \mathcal{A} \to \mathcal{A}/\!\sim$. We call $\mathcal{A}/\!\sim$ the *separated collapse* of $\mathcal{A}$. The separated collapse of $\mathcal{A}^\perp$ is known as the *extensional collapse* of $\mathcal{A}$.

Proposition 5.21 $\mathbf{Chu}_{sep}$ *is a reflective subcategory of* $\mathbf{Chu(Set,}\ K\mathbf{)}$*, i.e. the inclusion* $\mathbf{Chu}_{sep} \hookrightarrow \mathbf{Chu}(\mathbf{Set}, K)$ *has a left adjoint* L*. Dually* $\mathbf{Chu}_{ex}$ *is coreflective, with coreflector* R*. Thus we obtain a polarized category model of* MALLP*:*

$$\begin{tikzcd} \mathbf{Chu}_{sep} \arrow[r, hook, bend right, "\mathsf{Inj}"'] \arrow[r, phantom, "\perp"] & \mathbf{Chu(Set,}\ K\mathbf{)} \arrow[l, bend right, "\mathsf{L}"'] \arrow[r, bend right, "\mathsf{R}"'] \arrow[r, phantom, "\perp"] & \mathbf{Chu}_{ex} \arrow[l, hook', bend right, "\mathsf{Inj}"'] \end{tikzcd}$$

Proof. Define $\mathsf{L}(\mathcal{A}) = \mathcal{A}/\!\sim$. Given a morphism $(f,g) : \mathcal{A} \to \mathcal{B}$, define $\mathsf{L}(f,g) = (f/\!\sim, g) : \mathcal{A}/\!\sim \longrightarrow \mathcal{B}/\!\sim$ where $f/\!\sim : A/\!\sim \longrightarrow B/\!\sim$ is the map on equivalence classes $[a] \mapsto [f(a)]$. $f/\!\sim$ is well-defined; if $a_1 \sim a_2$, then for all $b \in Y$ $R(a_1, g(b)) = R(a_2, g(b))$, which means $S(f(a_1), b) = S(f(a_2), b)$, hence $f(a_1) \sim f(a_2)$. Then it is easy to check that the canonical quotient morphisms $\eta_\mathcal{A} := (\nu, id_X) : \mathcal{A} \to \uparrow \mathcal{A}$ determine the unit $\eta : Id_{\mathbf{Chu}_{sep}} \to \mathsf{Inj}\ \mathsf{L}$ for the desired adjunction $\mathsf{L} \dashv \mathsf{Inj}$. $\square$

Lemma 5.22 ($\mathbf{Chu}_{ex}$ and $\mathbf{Chu}_{sep}$ are respectively positive and negative.) $\mathbf{Chu}_{ex}$ *is closed under* $\otimes$ *and* $\oplus$. *Dually,* $\mathbf{Chu}_{sep}$ *is closed under* $⅋$ *and* $\&$.

Proof. Given $\mathcal{A}$ and $\mathcal{B}$ from $\mathbf{Chu}_{ex}$, it is easy to check that $\mathcal{A} \otimes \mathcal{B}$ has no repeated columns as follows: for $(h, k), (h', k') \in hom(\mathcal{A}, \mathcal{B}^{\perp})$, suppose $T(\langle a, b\rangle, (h, k)) = T(\langle a, b\rangle, (h', k'))$ for all $\langle a, b\rangle \in A \times B$. From the definition of T for tensor, this means $R(a, k(b)) = R(a, k'(b))$ for all $a \in A$ and $b \in B$. Since $\mathcal{A}$ is extensional, R has no repeated columns. Thus $k(b) = k'(b)$ for all $b \in B$, so $k = k'$. In the same way, starting from the other definition of $T(\langle a, b\rangle, (h, k)) = S(b, h(a))$, we again obtain $h = h'$. It is more direct to check that $\mathbf{Chu}_{ex}$ is closed under $\oplus$. $\square$

From the above, we obtain the following proposition by setting $\mathcal{C}_- := \mathbf{Chu}_{sep}$ and $\mathcal{C}_+ := \mathbf{Chu}_{ex}$, so that $\mathsf{L} :=\uparrow$ and $\mathsf{R} :=\downarrow$;

Proposition 5.23 $\mathbf{Chu}_{+,-}$ *is a polarized category model for* MALLP.

Remark 5.24 (Functors $\uparrow$ and $\downarrow$ are non trivial in $\mathbf{Chu}_{+,-}$) In $\mathbf{Chu}_{+,-}$, the functors $\uparrow$ and $\downarrow$ act non-trivially on morphisms in general. However on the intersection of the two subcategories $\mathbf{Chu}_{sep}$ and $\mathbf{Chu}_{ex}$, the functors $\uparrow$ and $\downarrow$ act as the identity on morphisms, which causes the following:

$$\downarrow\uparrow\mathcal{C} = \mathsf{Fix}\epsilon \quad \text{and} \quad \uparrow\downarrow\mathcal{C} = \mathsf{Fix}\eta.$$

This validates the condition (16) of Remark 8.1, thus $\mathbf{Chu}_{+,-}$ is not faithful. In particular even in this framework of $\mathbf{Chu}_{+,-}$ with nontriviality of $\uparrow$ and $\downarrow$, the interpretations of the two proofs of Example 2.2 collapse to be the same; i.e., both are interpreted by the identity $(id_A, id_X/\sim)$ on $\downarrow\mathcal{A} := (A, R/\sim, X/\sim)$, where $\sim$ denotes the equivalence relation determined by $Ker(\check{R})$. This non-discrimination arises because for $\mathcal{A} \in \mathbf{Chu}_{sep}$, $\downarrow\mathcal{A} \in \mathbf{Chu}_{sep} \cap \mathbf{Chu}_{ex}$, thus $\uparrow\downarrow\uparrow\mathcal{A}^{\perp} =\uparrow\mathcal{A}^{\perp}$ and $\uparrow\downarrow\mathcal{A} =\downarrow\mathcal{A}$, which makes the the type of each morphism $\vdash\uparrow\mathcal{A}^{\perp}, \downarrow\mathcal{A}$.

5.3 Double Gluing Categories $\mathbf{G}\mathcal{C}_{+,-}$ over $\mathcal{C}_{+,-}$ and Polarities

In this subsection we shall apply Hyland-Tan's double gluing construction [41] to an arbitrary polarized category $\mathcal{C}_{+,-}$ so as to yield again a categorical framework $\mathbf{G}\mathcal{C}_{+,-}$. We assume the reader is familiar with the definition of double gluing (we review the notions in Appendix 2).

Let $\mathcal{C}_{+,-}$ be a polarized $*$-autonomous category in the sense of Definition 3.1. For ease of reading we use the notation in our proof of Soundness (Proposition 4.2): let $G : \mathcal{C}_- \hookrightarrow \mathcal{C}$ be the inclusion Inj_-, and $F : \mathcal{C} \to \mathcal{C}_-$ be its left adjoint $\uparrow$.

Definition 5.25 (positive and negative subcategories of $\mathbf{G}\mathcal{C}$) The subcategory $\mathbf{G}\mathcal{C}_-$ (resp. $\mathbf{G}\mathcal{C}_+$) of $\mathbf{G}\mathcal{C}$ consists of objects $\mathcal{A}$ of $\mathbf{G}\mathcal{C}$ such that $U(\mathcal{A})$ are objects in $\mathcal{C}_-$ (resp. in $\mathcal{C}_+$).

Definition 5.26 (functors $\uparrow$) A functor $\uparrow: \mathbf{G}\mathcal{C} \longrightarrow \mathbf{G}\mathcal{C}_-$ is defined by
($\uparrow$ on objects of $\mathcal{C}$) For an object $\mathcal{A} = (A, \mathcal{A}_p, \mathcal{A}_{cp})$ in $\mathbf{G}\mathcal{C}$, we define;

$$\uparrow\mathcal{A} := (\uparrow A, (\uparrow\mathcal{A})_p, (\uparrow\mathcal{A})_{cp})$$

where

- $(\uparrow\mathcal{A})_p$ is defined to be a subset of a homset $\mathcal{C}(\mathbf{1},\uparrow A)$ obtained from every $\alpha : \mathbf{1} \longrightarrow A$ in $\mathcal{A}_p$ as follows: its arrows are of the form $F(\alpha)\circ\eta_{\mathbf{1}}$, where

$$\mathbf{1} \xrightarrow{\eta_{\mathbf{1}}} GF\mathbf{1} \xrightarrow{=} F\mathbf{1} \xrightarrow{F\alpha} \uparrow A$$

- $(\uparrow\mathcal{A})_{cp}$ is defined to be a subset of a homset $\mathcal{C}(\uparrow A, \perp)$ obtained from every $\alpha' : A \longrightarrow \perp$ in $\mathcal{A}_{cp}$ as follows: its arrows are of the form $\epsilon_\perp\circ F(\alpha')$, where

$$FA \xrightarrow{F\alpha'} F\perp \xrightarrow{=} FG\perp \xrightarrow{\epsilon_\perp} \perp$$

($\uparrow$ on morphisms of $\mathcal{C}$) For a morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$, the morphism $F(f)$ satisfies the points and copoints condition to be a morphism: $\uparrow f : \uparrow\mathcal{A} \longrightarrow \uparrow\mathcal{B}$ directly as follows:

- (point condition for $\uparrow f$) This is a condition that for all $F(\alpha)\circ\eta_1 \in (\uparrow\mathcal{A})_p$, it holds that $F(f)\circ F(\alpha)\circ\eta_1 \in (\uparrow\mathcal{B})_p$: First the point condition of f tells us that $f\circ\alpha = \beta$ for some $\beta \in \mathcal{B}_p$. Thus, $F(f)\circ F(\alpha)\circ\eta_{\mathbf{1}} = F(\beta)\circ\eta_{\mathbf{1}}$, which is an element of $(\uparrow\mathcal{B})_p$ from the definition.
- (copoint condition for $\uparrow f$) Dually to the above.

Proposition 5.27 (An adjunction $\uparrow\dashv \mathsf{Inj}\_$) *For every object $\mathcal{A} \in \mathbf{G}\mathcal{C}$ and $\mathcal{B} \in \mathbf{G}\mathcal{C}_-$, there is a natural isomorphism*

$$\mathbf{G}\mathcal{C}_-(F\mathcal{A}, \mathcal{B}) \cong \mathbf{G}\mathcal{C}(\mathcal{A}, G\mathcal{B})$$

where $F = \uparrow$ and $G = \mathsf{Inj}\_$.

Proof. We shall show an adjunction $\uparrow\dashv \mathsf{Inj}\_$ of $\mathcal{C}$ can be lifted up to $\mathbf{G}\mathcal{C}$ to retain point/copoint conditions.
($\Leftarrow$) Given $f : A \to GB$ from the R.H.S, we shall show that its adjunction $\epsilon_B \circ Ff$ in $\mathcal{C}$

$$FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$$

satisfies the point/copoint conditions of the L.H.S for $\mathbf{G}\mathcal{C}_-$:

(point-condition) For $f : A \to GB = B$, the point condition for R.H.S is

$$1 \xrightarrow{\forall\alpha \in \mathcal{A}_p} A \xrightarrow{f} B \quad \in \mathcal{B}_p.$$

The condition for the assertion is that the following morphism belongs to $\mathcal{B}_p$;

$$1 \xrightarrow{\eta_1} GF1 \xrightarrow{=} F1 \xrightarrow{F\alpha} FA \xrightarrow{Ff} FB \xrightarrow{=} FGB \xrightarrow{\epsilon_B} B$$

But the above two morphisms are equal by properties of the units and counits, [3]which implies the assertion.

[3]This is the same equation of morphisms as the last diagram of Soundness 4.2

(copoint-condition) For $A \to GB = B$, the copoint condition for the R.H.S is

$$A \xrightarrow{\ \ f\ \ } B \xrightarrow{\forall \beta \in \mathcal{B}_{cp}} \perp \quad \in \mathcal{A}_{cp}.$$

The condition for the assertion is that the following top-most horizontal arrow belongs to $(\uparrow \mathcal{A})_{cp}$ for all $\beta \in \mathcal{B}_{cp}$, which is derived from the commutativity of the following diagram by the naturality of ϵ and the above condition:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] & FB \arrow[r, "="] \arrow[drr, "F\beta"'] & FGB \arrow[r, "\epsilon_B"] & B \arrow[r, "\beta"] & \perp \\
& & & F\perp = FG\perp \arrow[ur, "\epsilon_\perp"'] &
\end{tikzcd}$$

($\Rightarrow$) Given $g : FA \to B$ from the L.H.S, we shall show that its adjunction $\eta_A \circ g$ in $\mathcal{C}$

$$A \xrightarrow{\ \eta_A\ } GFA \xrightarrow{Gg = g} GB$$

satisfies the point condition and the copoint condition for the R.H.S.

(point-condition)
For $g : FA \to B$, the point condition for the L.H.S assures that the top-most horizontal arrow of the following diagram belongs to $\mathcal{B}_p$ for all $\alpha \in \mathcal{A}_p$. The condition for the assertion is directly derived from the commutativity of the diagram which is naturality of η;

$$\begin{tikzcd}
1 \arrow[r, "\eta_1"] \arrow[dr, "\alpha"'] & F1 \arrow[r, "F(\alpha)"] & F(A) \arrow[r, "g"] & B \\
& A \arrow[ur, "\eta_A"'] & &
\end{tikzcd}$$

(copoint-condition)
For $g : FA \to B$, the copoint condition for L.H.S is described by commutativity of the upper triangle of the following diagram; i.e., $\forall\, \beta \in \mathcal{B}_{cp}\, \exists\, \alpha \in \mathcal{A}_{cp}\ \ \beta \circ g = \epsilon_\perp \circ F(\alpha)$.

$$\begin{tikzcd}
& F\perp = FG\perp \arrow[dr, "\epsilon_\perp"] & \\
FA \arrow[ur, "F\alpha"] \arrow[r, "g"] & B \arrow[r, "\beta"] & \perp \\
& A \arrow[ul, "\eta_A"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

The condition for the assertion is that $\beta \circ g \circ \eta_A$ belongs to $\mathcal{A}_{cp}$. Since $\epsilon_\perp \circ F\alpha \circ \eta_A = \alpha$ [4], as shown by commutativity of the outermost square of the above diagram, the condition is derived from the commutativity of the lower triangle. □

Proposition 5.28 (Polarized Category $\mathbf{G}\mathcal{C}_{+,-}$) *$\mathbf{G}\mathcal{C}_{+,-}$ is a polarized category whenever $\mathcal{C}_{+,-}$ is.*

Example 5.29 $\mathbf{GHCoh}_{+,-}$ is a polarized category built from $\mathbf{HCoh}_{+,-}$ as studied in Section 5.1. $\mathbf{GHCoh}_{+,-}$ does not validate the $\mathsf{Mix_p}$ inference (see Remark 7.18 below). One goal of this paper is to prove MLLP full completeness in a dinatural framework over $\mathbf{GHCoh}_{+,-}$ (cf. Theorem 7.21).

5.4 Iterated Double Gluing $\mathbf{G}^2\mathcal{C}$ and Polarities $(\mathbf{G}^2\mathcal{C})_{+,-}$

In this subsection, we show that a simple notion of polarity arises through iterations $\mathbf{G}^2\mathcal{C}$ of Hyland-Tan's double gluing construction over an arbitrary $*$-autonomous category $\mathcal{C}$. We assume the reader is familiar with the definition of double gluing (we review the notions in Appendix 2).

An object $\mathcal{A}$ of an iterated double gluing category $\mathbf{G}^2\mathcal{C} = \mathbf{G}(\mathbf{G}\mathcal{C})$ is of the following form, by the definition of double gluing:

$$\mathcal{A} := (U\mathcal{A}, \mathcal{A}_p, \mathcal{A}_{cp})$$

where $U\mathcal{A}$ is an object of $\mathbf{G}\mathcal{C}$ and

$$\begin{cases} \mathcal{A}_p \subseteq \mathbf{G}\mathcal{C}(\mathbf{1}, U\mathcal{A}) \cong (U\mathcal{A})_p \\ \mathcal{A}_{cp} \subseteq \mathbf{G}\mathcal{C}(U\mathcal{A}, \perp) \cong (U\mathcal{A})_{cp} \end{cases}$$

Again by the definition of double gluing $\mathbf{G}\mathcal{C}$, $U\mathcal{A}$ is of the following form:

$$U\mathcal{A} := (U^2\mathcal{A}, (U\mathcal{A})_p, (U\mathcal{A})_{cp})$$

where $U^2\mathcal{A}$ is an object of $\mathcal{C}$ and

$$\begin{cases} (U\mathcal{A})_p \subseteq \mathcal{C}(\mathbf{1}, U^2\mathcal{A}) \\ (U\mathcal{A})_{cp} \subseteq \mathcal{C}(U^2\mathcal{A}, \perp) \end{cases}$$

From the above, any object $\mathcal{A}$ of $\mathbf{G}^2\mathcal{C}$ is written as a triple as follows:

$$\mathcal{A} := (U^2\mathcal{A},\ \mathcal{A}_p \subseteq (U\mathcal{A})_p,\ \mathcal{A}_{cp} \subseteq (U\mathcal{A})_{cp})$$

where $U^2\mathcal{A}$ is an object of $\mathcal{C}$, $(U\mathcal{A})_p \subseteq \mathcal{C}(\mathbf{1}, U^2\mathcal{A})$ and $(U\mathcal{A})_{cp} \subseteq \mathcal{C}(U^2\mathcal{A}, \perp)$.

More generally an object $\mathcal{A}$ of $\mathbf{G^n}\mathcal{C}$ is written as a triple as follows:

$$\mathcal{A} := (U^n\mathcal{A},\ \mathcal{A}_p \subseteq (U\mathcal{A})_p \subseteq \cdots \subseteq (U^{n-1}\mathcal{A})_p,\ \mathcal{A}_{cp} \subseteq (U\mathcal{A})_{cp} \subseteq \cdots \subseteq (U^{n-1}\mathcal{A})_{cp})$$

where the first element is an object of $\mathcal{C}$ and the second (resp. third) element is an increasing sequence of length n of subsets of the homset $\mathcal{C}(\mathbf{1}, U^n\mathcal{A})$ (resp. $\mathcal{C}(U^n\mathcal{A}^\perp, \perp)$).

$$
\begin{tikzcd}[arrows=dash, column sep=small]
 & & & & & & \mathcal{A} \arrow[dl] \arrow[d] \arrow[dr] & \\
 & & & & & U\mathcal{A} \arrow[dl] \arrow[d] \arrow[dr] & \mathcal{A}_p & \mathcal{A}_{cp} \\
 & & & & U^2\mathcal{A} & (U\mathcal{A})_p & (U\mathcal{A})_{cp} & \\
 & & & \iddots & & & & \\
 & & & U^k\mathcal{A} \arrow[dl] \arrow[d] \arrow[dr] & & & & \\
 & & U^{k+1}\mathcal{A} & (U^k\mathcal{A})_p & (U^k\mathcal{A})_{cp} & & & \\
 & \iddots & & & & & & \\
 & U^{n-1}\mathcal{A} \arrow[dl] \arrow[d] \arrow[dr] & & & & & & \\
U^n\mathcal{A} & (U^{n-1}\mathcal{A})_p & (U^{n-1}\mathcal{A})_{cp} & & & & &
\end{tikzcd}
$$

Figure 1: An object $\mathcal{A}$ of $\mathbf{G^n}\mathcal{C}$

In the following Figure 1 we represent an object of $\mathbf{G^n}\mathcal{C}$ by means of a tree using 3-tuples $U^k\mathcal{A} = (U^{k+1}\mathcal{A}, (U^k\mathcal{A})_p, (U^k\mathcal{A})_{cp}) \in \mathbf{G^{n-k}}\mathcal{C}$ as the (leftmost) ternary nodes.

The morphisms from $\mathcal{A}$ to $\mathcal{B}$ in $\mathbf{G^n}\mathcal{C}$ are morphisms f from $U^n\mathcal{A}$ and $U^n\mathcal{B}$ in $\mathcal{C}$ such that the following conditions with $0 \leq k < m$ hold:

(k-th point condition:) $\quad \forall \alpha \in (U^k\mathcal{A})_p \ [\alpha]f \in (U^k\mathcal{B})_p$

(k-th copoint condition:) $\quad \forall \beta \in (U^k\mathcal{B})_{cp} \ f[\beta] \in (U^k\mathcal{A})_{cp}$

We wish to discuss how positive and negative subcategories arise quite simply in the framework of the iterated double gluing category $\mathbf{G}^2\mathcal{C}$.

Definition 5.30 (positive and negative subcategories of $\mathbf{G}^2\mathcal{C}$)

- $(\mathbf{G}^2\mathcal{C})_+$ is a subcategory of $\mathbf{G}^2\mathcal{C}$ consisting of objects $\mathcal{A}$ satisfying $\mathcal{A}_p = (U\mathcal{A})_p$.
- Dually, $(\mathbf{G}^2\mathcal{C})_-$ is a subcategory of $\mathbf{G}^2\mathcal{C}$ consisting of objects $\mathcal{A}$ satisfying $\mathcal{A}_{cp} = (U\mathcal{A})_{cp}$.

The definition yields that $(\mathbf{G}^2\mathcal{C})_+$ (resp. $(\mathbf{G}^2\mathcal{C})_-$) is positive (resp. negative) in the following sense.

Proposition 5.31 *$(\mathbf{G}^2\mathcal{C})_+$ (resp. $(\mathbf{G}^2\mathcal{C})_-$) is closed under $\otimes$ and $\oplus$ (resp. $\parr$ and &).*

Proof. In the double gluing construction, points (resp. copoints) of $\otimes$ (resp. $\parr$) are constructed componentwise from points (resp. copoints) of each component (cf. Appendix 2 for $*$-autonomy in double gluing categories). Hence closedness of $\otimes$ (resp. $\parr$) in $(\mathbf{G}^2\mathcal{C})_+$ (resp. $(\mathbf{G}^2\mathcal{C})_-$) is easily derived. Closedness under additives is more direct. $\square$

Next we shall define functors $\uparrow$ and $\downarrow$ from $\mathbf{G}^2\mathcal{C}$ to the subcategories $(\mathbf{G}^2\mathcal{C})_-$ and $(\mathbf{G}^2\mathcal{C})_+$, respectively.

Definition 5.32 (functors $\uparrow$ and $\downarrow$) The functors $\uparrow\colon \mathcal{C} \to \mathcal{C}_-$ and $\downarrow\colon \mathcal{C} \to \mathcal{C}_+$ are defined by the following data:

[4]Again, this is the same equation of morphisms as the last diagram of Soundness 4.2

(On objects) For an object $\mathcal{A} = (U^2\mathcal{A},\ \mathcal{A}_p \subseteq (U\mathcal{A})_p,\ \mathcal{A}_{cp} \subseteq (U\mathcal{A})_{cp})$ of $\mathbf{G}^2\mathcal{C}$, we define

$$\uparrow\mathcal{A} := (U^2\mathcal{A},\ \mathcal{A}_p \subseteq (U\mathcal{A})_p,\ \mathcal{A}_{cp} = \mathcal{A}_{cp})$$

That is

$$\begin{cases} (\uparrow\mathcal{A})_p := \mathcal{A}_p \text{ and } (U(\uparrow\mathcal{A}))_p := (U\mathcal{A})_p \\ (\uparrow\mathcal{A})_{cp} := (U(\uparrow\mathcal{A}))_{cp} := \mathcal{A}_{cp} \end{cases}$$

Dually, we define

$$\downarrow\mathcal{A} := (U^2\mathcal{A},\ \mathcal{A}_p = \mathcal{A}_p,\ \mathcal{A}_{cp} \subseteq (U\mathcal{A})_{cp})$$

That is

$$\begin{cases} (\downarrow\mathcal{A})_p := (U(\downarrow\mathcal{A}))_p := \mathcal{A}_p \\ (\downarrow\mathcal{A})_{cp} := \mathcal{A}_{cp} \text{ and } (U(\downarrow\mathcal{A}))_{cp} := (U\mathcal{A})_{cp} \end{cases}$$

(On morphisms) $\uparrow$ and $\downarrow$ act on morphisms as the identity; i.e., for $f \in \mathbf{G}^2\mathcal{C}(\uparrow\mathcal{A}, \uparrow\mathcal{B})$, we define $\uparrow f \in \mathbf{G}^2\mathcal{C}(\uparrow\mathcal{A}, \uparrow\mathcal{B})$ and $\downarrow f \in \mathbf{G}^2\mathcal{C}(\downarrow\mathcal{A}, \downarrow\mathcal{B})$ by

$$\uparrow f := f := \downarrow f$$

It can be directly checked that the above is well-defined; i.e., for $f \in \mathbf{G}^2\mathcal{C}(\mathcal{A}, \mathcal{B})$, it holds that $f \in \mathbf{G}^2\mathcal{C}(\uparrow\mathcal{A}, \uparrow\mathcal{B})$ and $f \in \mathbf{G}^2\mathcal{C}(\downarrow\mathcal{A}, \downarrow\mathcal{B})$.

Now we have adjunctions on the functors $\uparrow$ and $\downarrow$ defined above.

Proposition 5.33 (Adjunctions $\mathsf{Inj}_+ \dashv \downarrow$ and $\uparrow \dashv \mathsf{Inj}_-$)
For all objects $\mathcal{A} \in (\mathbf{G}^2\mathcal{C})_+$ and $\mathcal{B} \in \mathbf{G}^2\mathcal{C}$,

$$\mathbf{G}^2\mathcal{C}(\mathcal{A}, \mathcal{B}) = (\mathbf{G}^2\mathcal{C})_+(\mathcal{A}, \downarrow\mathcal{B})$$

Dually for all objects $\mathcal{A} \in \mathbf{G}^2\mathcal{C}$ and $\mathcal{B} \in (\mathbf{G}^2\mathcal{C})_-$,

$$\mathbf{G}^2\mathcal{C}(\mathcal{A}, \mathcal{B}) = (\mathbf{G}^2\mathcal{C})_-(\uparrow\mathcal{A}, \mathcal{B})$$

Proof. Straightforward as follows: for the first adjunction, for every morphism whose domain is positive $\mathcal{A}$, the first and the second point conditions for f collapse to be the same since $\mathcal{A}_p = (U\mathcal{A})_p$. Dually for the second adjunction. □

From the above, we have

Proposition 5.34 *$(\mathbf{G}^2\mathcal{C})_{+,-}$ is a polarized category.*

Finally, we end this section on examples of models with the following observation, which follows from a detailed examination of their structure.

Remark 5.35 (Strictness of the structure of models) The above 4 classes of models all have strict polarized structure, not just up-to-isomorphism. That is, the $*$-autonomous de Morgan structure is strict and the functors $\uparrow$ and $\downarrow$ are (strictly) de Morgan dual as well as being strict monoidal functors (in the case of double gluing categories $\mathbf{G}\mathcal{C}_{+,-}$, this assumes that $\mathcal{C}_{+,-}$ is strictly polarized and the appropriate definition of points and copoints is used.)

6 Dinatural Frameworks

6.1 Polarized Functoriality

In this subsection we shall introduce a polarized version of functoriality for our polarized category $\mathcal{C}_{+,-}$. A *polarized multivariant functor* is a functor $F : (\mathcal{C}_{-}^{op})^n \times (\mathcal{C}_{-})^n \to \mathcal{C}$. Recall that $(\mathcal{C}_{-})^{op} \cong \mathcal{C}_{+}$, so equivalently a polarized multivariant functor is a functor $F : (\mathcal{C}_{+})^n \times (\mathcal{C}_{-})^n \to \mathcal{C}$. A polarized multivariant functor is called *positive* (resp. *negative*) if its range is contained in $\mathcal{C}_{+}$ (resp. $\mathcal{C}_{-}$).

We extend functorial polymorphism for linear logic (see [11, 12]) to the polarized case by interpreting polarized formulas as usual but using polarized functors, (note that variables are positive, thus interpreted as covariant projections as always), with the following added rules obtained by using the adjoint functors $\Uparrow \dashv \Downarrow$ between $\mathcal{C}_{-}$ and $\mathcal{C}_{+}$ (recall Diagram (7) and the notation below it), as follows:

Let $F : (\mathcal{C}_{+})^n \times (\mathcal{C}_{-})^n \to \mathcal{C}$. Define

- $\uparrow F = (\mathcal{C}_{+})^n \times (\mathcal{C}_{-})^n \xrightarrow{F} \mathcal{C} \xrightarrow{\uparrow} \mathcal{C}_{-}$
- $\downarrow F = (\mathcal{C}_{+})^n \times (\mathcal{C}_{-})^n \xrightarrow{F} \mathcal{C} \xrightarrow{\downarrow} \mathcal{C}_{+}$

A *polarized dinatural transformation* between polarized functors is a family of morphisms in $\mathcal{C}_{-}$ (or dually in $\mathcal{C}_{+}$) or in a module in $\widehat{\mathcal{C}}$ satisfying the usual hexagonal condition for all $\vec{f} : \vec{A} \to \vec{B}$ in $(\mathcal{C}_{-})^n$ as follows:

Definition 6.1 (polarized dinatural transformation) For polarized multivariant functors $E, F : (\mathcal{C}_{-}^{op})^n \times (\mathcal{C}_{-})^n \to \mathcal{C}$, a *polarized dinat*

$$\theta \in pDinat\text{-}\mathcal{C}_{+,-}$$

is a family of morphisms $\theta_{\vec{A}}$ in $\mathcal{C}_{-}$ (or dually in $\mathcal{C}_{+}$) or in a module in $\widehat{\mathcal{C}}$

$$\theta := \{\theta_{\vec{A}} : E(\vec{A};\vec{A}) \to F(\vec{A};\vec{A}) \mid \vec{A} \in (\mathcal{C}_{-})^n\}$$

such that for all $\vec{f} : \vec{A} \to \vec{B}$ in $(\mathcal{C}_{-})^n$, the following hexagonal diagram commutes:

$$\begin{tikzcd}
 & E(\vec{A};\vec{A}) \arrow[r, "\theta_{\vec{A}}"] & F(\vec{A};\vec{A}) \arrow[dr, "F(\vec{A};\vec{f})"] & \\
E(\vec{B};\vec{A}) \arrow[ur, "E(\vec{f};\vec{A})"] \arrow[dr, "E(\vec{B};\vec{f})"'] & & & F(\vec{A};\vec{B}) \\
 & E(\vec{B};\vec{B}) \arrow[r, "\theta_{\vec{B}}"] & F(\vec{B};\vec{B}) \arrow[ur, "F(\vec{f};\vec{B})"'] &
\end{tikzcd}$$

In usual non-polarized functorial frameworks, we sometimes introduce the notion of *restricting dinaturals to a subcategory* by allowing variables and morphisms in multivariant functors to range over the subcategory (see [6]). This framework is used in Hamano [23] for multiplicative full completeness. In the polarized case, we show this provides an important bridge between the polarized and the usual notion of dinatural.

Definition 6.2 ($Dinat_{\mathcal{D}}$-$\mathcal{C}$) Let $\mathcal{C}$ be a $*$-autonomous category and $\mathcal{D}$ a subcategory. For multivariant functors $E, F : (\mathcal{D}^{op})^n \times \mathcal{D}^n \to \mathcal{C}$, define *dinaturality with respect to* $\mathcal{D}$ as follows: a family $\theta \in Dinat_{\mathcal{D}}\text{-}\mathcal{C}(E, F)$ is a family of $\mathcal{C}$-morphisms $\theta := \{\theta_{\vec{A}} : E(\vec{A};\vec{A}) \to F(\vec{A};\vec{A}) \mid \vec{A} \in \mathcal{D}^n\}$ such that for all $\vec{f} : \vec{A} \to \vec{B}$ in $\mathcal{D}^n$, the following hexagonal diagram commutes:

$$\begin{tikzcd}
& E(\vec{A};\vec{A}) \arrow[r, "\theta_{\vec{A}}"] & F(\vec{A};\vec{A}) \arrow[dr, "F(\vec{A};\vec{f})"] & \\
E(\vec{B};\vec{A}) \arrow[ur, "E(\vec{f};\vec{A})"] \arrow[dr, "E(\vec{B};\vec{f})"'] & & & F(\vec{A};\vec{B}) \\
& E(\vec{B};\vec{B}) \arrow[r, "\theta_{\vec{B}}"] & F(\vec{B};\vec{B}) \arrow[ur, "F(\vec{f};\vec{B})"'] &
\end{tikzcd}$$

If $\theta \in Dinat\text{-}\mathcal{C}(E, F)$, then for any subcategory $\mathcal{D}$ of $\mathcal{C}$ (as above), we obtain an induced family $\theta^{\mathcal{D}}$ of morphisms by restricting θ to $\mathcal{D}$; i.e. define $\theta^{\mathcal{D}} := \{\theta_{\vec{D}} : E(\vec{D},\vec{D}) \to F(\vec{D},\vec{D}) \mid \vec{D} \in \mathcal{D}^n\}$. Observe $\theta^{\mathcal{D}} \in Dinat_{\mathcal{D}}\text{-}\mathcal{C}(E, F)$, hence $\{\theta^{\mathcal{D}} \mid \theta \in Dinat\text{-}\mathcal{C}(E, F)\} \subseteq Dinat_{\mathcal{D}}\text{-}\mathcal{C}(E, F)$, so there is an induced canonical map $Dinat\text{-}\mathcal{C}(E, F) \to Dinat_{\mathcal{D}}\text{-}\mathcal{C}(E, F)$ given by $\theta \mapsto \theta^{\mathcal{D}}$.

In our Full Completeness Theorems (in Sections 7.2 and 7.3 below) we will restrict ourselves to dinats between *definable* functors.

Proposition 6.3 (From Polarized to NonPolarized) *There is a canonical map*

$$U : \quad pDinat\text{-}\mathcal{C}_{+,-} \longrightarrow Dinat_{\mathtt{Fix}\epsilon}\text{-}\mathcal{C}$$

which is a canonical "depolarizing" map from a polarized category $\mathcal{C}_{+,-}$ to a $*$-autonomous category $\mathcal{C}$.

Proof. U is the restriction of a $\mathcal{C}_-$-indexed dinatural family ρ of $\mathcal{C}_{+,-}$-morphisms from the L.H.S. into a subcategory $\mathtt{Fix}\epsilon$ of $\mathcal{C}_-$ as follows:

$$U(\rho) := \{\rho_{\vec{A}} : E(\vec{A};\vec{A}) \longrightarrow F(\vec{A};\vec{A}) \mid \vec{A} \in \mathtt{Fix}\epsilon\}$$

Since in a subcategory $\mathtt{Fix}\epsilon$, $\uparrow$ and $\downarrow$ act as the identity on objects (hence on morphisms), it holds that $G(\vec{A};\vec{A}) \cong |G|\,(\vec{A};\vec{A})$ for every polarized definable multivariant functor $G(\vec{X};\vec{X})$, where $|G|$ denotes G by erasing $\uparrow$ and $\downarrow$. Thus we conclude that $U(\rho)$ is in the R.H.S. $\Box$

In particular, if $\mathcal{C}_{+,-}$ is $\mathbf{HCoh}_{+,-}$ in the above proposition, we have moreover the following:

Proposition 6.4 (From $\mathbf{HCoh}_{+,-}$ to Coh) *If $\mathcal{C}_{+,-}$ is $\mathbf{HCoh}_{+,-}$ in Proposition 6.3 above, then in this case U acts as an identity map between dinats as follows:*

$$U: \ pDinat\text{-}\mathbf{HCoh}_{+,-} \longrightarrow Dinat\text{-}\mathbf{Coh}$$

That is, U is a canonical "depolarizing" map satisfying $U(\rho) = \rho$.

Proof. First, we shall show that the target of U is $Dinat$-**Coh**. This is because $\mathtt{Fix}\epsilon$ is isomorphic to the category of coherent spaces **Coh** (see Proposition 5.15), which happens to be $*$-autonomous, together with the fact that

$$Dinat_{\mathbf{Coh}}\mathbf{HCoh} \cong Dinat\text{-}\mathbf{Coh} \ .$$

Second, we must show that U acts as the identity on dinats, i.e. that $U(\rho) = \rho$. This means that for all $\vec{A} \in \mathbf{HCoh}_-$, $\rho_{\vec{A}}$ and $U(\rho)_{U\vec{A}}$ are the same relation. This holds because $\mathtt{Fix}\epsilon$ coincides with $\uparrow\downarrow\mathbf{HCoh}$ (see Proposition 5.15), hence the target of U is $Dinat_{\uparrow\downarrow\mathbf{HCoh}}$-$\mathbf{HCoh}$. On the other hand, there is a canonical morphism for every object $N \in \mathbf{HCoh}_-$

$$\uparrow\downarrow N \longrightarrow N,$$

which is the counit (9) of the adjunction $\Uparrow\dashv\Downarrow$ and is the identity of $|N|$ in the case of $\mathbf{HCoh}_{+,-}$ (see Propositions 5.11 and 5.12). Then the hexagonal diagrams of Definition 6.1 with respect to these morphisms yield that an $\uparrow\downarrow\mathbf{HCoh}$ indexed family of morphisms is enough to determine a $\mathbf{HCoh}_-$ indexed family of morphisms for a polarized dinat of L.H.S. $\square$

Example 6.5 We illustrate the above Proposition:

1. $Dinat$-**Coh** is fully complete for $\mathsf{MLL} + \mathsf{Mix}$, which was proved by Tan [41] (see also Proposition 3.7 of [10]). Hence in Proposition 6.4 if a dinat ρ is definable from an MLLP formula, its depolarization $U(\rho)$ is a denotation of an $\mathsf{MLL} + \mathsf{Mix}$ proof.
2. In Proposition 7.25 in Subsection 7.4 below it is shown how to define a polarized Gustave function, with an associated dinatural family R which is definable from a MALLP formula. Then its depolarization $U(R)$ is the usual dinat of a 3-ary Gustave function (cf. Proposition 2.11 of [10]). This will be used in Subsection 7.4 below to show that $\mathbf{HCoh}_{+,-}$ is not fully complete for MALLP.

Proposition 6.4 above is critical to our main Full Completeness Theorem for $\mathsf{MLLP}+\mathsf{Mix_p}$ (see Theorem 7.15 below).

7 Completeness Theorems

7.1 Completeness, Full Completeness and Polarized Softness

We now consider completeness theorems for our formalism. Here "completeness" can mean one of two things: "completeness with respect to provability" (called "weakly full completeness" in the terminology of Harnik and Makkai's paper on Läuchli semantics. [26]), versus "completeness with respect to proofs" (often called *full completeness*, see Blute and Scott's Linear Läuchli semantics [11]).

Remark 7.1 Let $\mathcal{L}$ be a theory, i.e. a language with associated logical and nonlogical axioms. In categorical logic, we think of the syntax (and axioms) of $\mathcal{L}$ as forming a *free, structured category $\mathcal{F}$ given by generators and relations.* Here the objects of $\mathcal{F}$ are the formulas and a morphism from A to B is an equivalence class of proofs of the entailment $A \vdash B$, modulo some appropriate notion of "equivalence of proofs" (see [26, 33]). We often identify $\mathcal{L}$ with this free category $\mathcal{F}$. In this case, to say that $Hom_{\mathcal{F}}(A, B) \neq \emptyset$ simply means that the sequent $A \vdash B$ is provable in $\mathcal{L}$. From this viewpoint, the usual logicians' notion of *interpretation* of a theory $\mathcal{L}$ in a model category $\mathcal{M}$ arises immediately from the freeness of $\mathcal{F}$ ([33]). Namely, with respect to some interpretation of the basic generators, an interpretation $[\![-]\!] : \mathcal{L} \to \mathcal{M}$ is simply the (unique) structure-preserving functor $[\![-]\!] : \mathcal{F} \to \mathcal{M}$ guaranteed by the freeness of the category $\mathcal{F}$ (i.e. the syntax $\mathcal{L}$) in some appropriate category of models.

Definition 7.2 Let $\mathcal{M}$ be a categorical model for a language $(\mathcal{L}, \vdash)$. An interpretation $\mathcal{L} \xrightarrow{[\![-]\!]} \mathcal{M}$ is *weakly full* if $Hom_{\mathcal{M}}([\![A]\!], [\![B]\!]) \neq \emptyset$ implies $A \vdash B$ is provable in $\mathcal{L}$, i.e. $[\![-]\!]$ is a surjective function from provable sequents to the homsets of $\mathcal{M}$. $\mathcal{M}$ is *weakly fully complete* (w.f.c) if the canonical (free) interpretation of $\mathcal{L}$ in $\mathcal{M}$ (wrt an interpretation of the generators) is weakly full. An interpretation is *fully complete* (f.c.) if $[\![-]\!]$ is a surjective map from proofs of sequents to the homsets of $\mathcal{M}$. This means: every morphism in $Hom_{\mathcal{M}}([\![A]\!], [\![B]\!])$ is the image of some proof of a sequent $A \vdash B$ in $\mathcal{L}$. Finally, an interpretation is *faithful* if $[\![-]\!]$ is injective on the set of proofs (with respect to some notion of equality of proofs in $\mathcal{L}$).

In the case of $\mathcal{L} = \mathsf{MALLP}$ and $\mathcal{M} = \mathcal{C}_{+,-}$ in the definitions above, we interpret them as follows. We first note that any sequent $A \vdash B$ in $\mathcal{L}$ is assumed to be in a legal pattern (as in Remark 3.2). Then $Hom_{\mathcal{M}}([\![A]\!], [\![B]\!])$ is equal to $\mathcal{C}([\![A]\!], [\![B]\!])$ for any such legal pattern, since $\mathcal{C}_+$ and $\mathcal{C}_-$ are full subcategories of $\mathcal{C}$ and the module $\widehat{\mathcal{C}}$ is given by the usual hom functor of the ambient $\mathcal{C}$. Second, we interpret weak fullness as follows, based upon the form of MALLP sequents. We say $\mathcal{C}_{+,-}$ is *weakly fully complete* (w.f.c.) if:

1. If $\mathcal{C}_-(N_1, N_2) \neq \emptyset$ then $N_1 \vdash N_2$ is provable in MALLP.

2. Dually, if $\mathcal{C}_+(P_1, P_2) \neq \emptyset$ then $P_1 \vdash P_2$ is provable in MALLP.

3. If $\widehat{\mathcal{C}}(P, N) \neq \emptyset$ then $P \vdash N$ is provable in MALLP.

Notice in all cases 1–3, we can uniformly replace $\mathcal{C}_-$, $\mathcal{C}_+$ and $\widehat{\mathcal{C}}$ by $\mathcal{C}$, for the same reasons that $Hom_{\mathcal{M}}([\![A]\!], [\![B]\!]) = \mathcal{C}([\![A]\!], [\![B]\!])$ above.

By results in Subsection 7.4 below (see also Example 6.5 (2) above), $\mathbf{HCoh}_{+,-}$ is not fully complete for MALLP. This suggests we consider the dinatural framework. However the problem is not so simple. First, $\mathbf{HCoh}_{+,-}$ has a polarized Gustave function (see Proposition 7.25) and this function also lives in the (polarized) dinatural framework. Unfortunately, Gustave functions are incompatible with full completeness, as we show below (see Proposition 7.26). However such Gustave functions only exist in the additive case, i.e. for MALLP. Hence we shall restrict ourselves to MLLP in our discussions below.

7.1.1 Polarized $\uparrow$-Softness

Definition 7.3 (Polarized Softness of $\mathcal{C}_{+,-}$) A polarized category $\mathcal{C}_{+,-}$ is called *n-ary $\uparrow$-soft* for $n \geq 1$ if every module

$$\rho : \mathbf{1} \quad \longrightarrow \uparrow P_1 \parr \cdots \parr \uparrow P_i \parr \cdots \parr \uparrow P_n \quad \in \widehat{\mathcal{C}}$$

factors through some unit $\eta_i : P_i \longrightarrow \uparrow P_i \in \widehat{\mathcal{C}}$ of the adjunction (2) as follows, where $\rho' \in \mathcal{C}_-$:

$$\begin{tikzcd}
 & \uparrow P_1, \ldots, P_i, \ldots, \uparrow P_n \arrow[d, "\eta_i"] \\
\mathbf{1} \arrow[ur, dotted, "\rho'"] \arrow[r, "\rho"'] & \uparrow P_1, \ldots, \uparrow P_i, \ldots, \uparrow P_n
\end{tikzcd}$$

We note that (though $\downarrow$ is a reversible connective) $\uparrow$ is far from reversible in general. In this sense, polarized softness gives a partial reversibility of the $\uparrow$ connective.

Remark 7.4 (n-dimensional push out condition) In our framework of positive/negative subcategories $\mathcal{C}_+/\mathcal{C}_-$ of a $*$-autonomous category $\mathcal{C}$, the condition of polarized n-ary softness can be characterized by means of an n-dimensional weak pushout (cf. Joyal [31]). E.g., when $n = 3$ the condition is equivalent to the fact that the following cube is a 3-dimensional weak pushout, where each $\mathbf{D}$ denotes the functor $\mathcal{C}_i \times \mathcal{C}_j \times \mathcal{C}_k \to \mathbf{Set}$ for appropriate $i, j, k \in \{+, -\}$ defined by $\mathbf{D}(A, B, C) := \mathcal{C}(\mathbf{1}, A \parr B \parr C)$. Each arrow of the cube come from unit(s) $P \longrightarrow \uparrow P$ of the adjunction (2).

$$\begin{tikzcd}
\mathbf{D}(A,B,C) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbf{D}(\uparrow A,B,C) \arrow[dr] \arrow[dd] & \\
 & \mathbf{D}(A,\uparrow B,C) \arrow[rr] \arrow[dd] & & \mathbf{D}(\uparrow A,\uparrow B,C) \arrow[dd] \\
\mathbf{D}(A,B,\uparrow C) \arrow[rr] \arrow[dr] & & \mathbf{D}(\uparrow A,B,\uparrow C) \arrow[dr] & \\
 & \mathbf{D}(A,\uparrow B,\uparrow C) \arrow[rr] & & \mathbf{D}(\uparrow A,\uparrow B,\uparrow C)
\end{tikzcd}$$

We observe that originally Joyal required the above diagram to be a pushout, not just a weak pushout. The weak notion suffices for our purposes here, and corresponds closer to the syntax, as in the following remark.

Remark 7.5 (Necessity of Softness for Full Completeness) $\uparrow$-Softness is a necessary condition for a MALLP full completeness theorem. First, observe that the syntax is "soft" in the following sense: if we consider a cut-free proof of a sequent as representing a morphism, say $\downarrow A_1 \otimes \cdots \otimes \downarrow A_{m-1} \vdash \uparrow A_m \parr \cdots \parr \uparrow A_n$, it must end with either a $\downarrow$-left, or a $\uparrow$-right rule[5]. This guarantees softness for any fully complete categorical model as follows: by abuse

[5] Strictly speaking, proof theorists would replace the $\otimes$'s on the left side and $\parr$'s on the right side of the sequent by commas.

of notation, if in a model we have a morphism $\downarrow A_1 \otimes \cdots \otimes \downarrow A_{m-1} \to \uparrow A_m \parr \cdots \parr \uparrow A_n$, by fullness this arises from a (cut-free) proof of a sequent as above. Hence by the softness of the syntax, the proof factors through either a counit $\downarrow A \to A$ of the adjunction (1) on the left or a unit $A \to \uparrow A$ of the adjunction (2) on the right. By the Soundness Theorem, this factorization is transformed (by the interpretation of the syntax in the model) into a factorization of the original morphism.

Remark 7.6 (Polarized Proof-Structures are Polarized Soft) Let us consider the proof-structure counterpart of the notion of polarized softness. One may observe directly that Laurent's MALLP proof-structures [35] are soft; i.e., if a polarized MALLP proof-structure Θ has conclusions $\uparrow P_1, \ldots, \uparrow P_n$, then some $\uparrow$-link with conclusion $\uparrow P_i$ can be removed to yield a proof structure with conclusions $\uparrow P_1, \ldots, P_i, \ldots, \uparrow P_n$. In the graph-theoretical framework of proof-structures (not only proof-nets), this polarized softness holds more directly than in general categorical frameworks, since it is graph theoretically straightforward to check that among the $\uparrow$-links of conclusions, there exists at least one which is not among the (auxiliary) conclusions of $\downarrow$-boxes (hence it is removable). Moreover this polarized softness is still quite trivial compared to usual additive $\oplus$-softness for nonpolarized MALL proof structures, which was studied in Hamano [24].

In the next subsection we study an appropriate *multiplicative* version of polarized softness in the concrete framework of *pDinat*-$\mathbf{HCoh}_{+,-}$. This is done by making a connection between this property and multiplicative full completeness of *Dinat*-$\mathbf{Coh}$ (cf. the above Example 6.5 (1)). By using softness of *pDinat*-$\mathbf{HCoh}_{+,-}$, we shall obtain MLLP full completeness.

7.1.2 An aside: $\oplus$-Softness in Polarized Categories

In this brief subsubsection, we discuss in our polarized case what becomes of the usual $\oplus$-softness in MALL studied in Blute-Hamano-Scott [10]. In our polarized category, because of the constraint of focalization, the outermost $\oplus$ connective always occurs focused; hence there is at most one such connective. So in the polarized case, the usual n-ary softness for $\oplus$ (cf. Definition 2.7 of [10]) for a natural number n is reduced to the following very special form of unary-softness.

Definition 7.7 (focalized unary-softness for $\oplus$ in MALLP) Every morphism of the form

$$\mathbf{1} \longrightarrow P_1 \oplus P_2, \uparrow \mathcal{P}$$

where $\uparrow \mathcal{P}$ denotes $\uparrow Q_1, \ldots, \uparrow Q_n, \mathcal{X}$ with $\mathcal{X}$ variables, factors through a coproduct injection of the focalized $\oplus$.

We call this softness *unary-softness* in analogy with the MALL case (cf. Definition 2.7 of [10]). Hence such a factorized $\oplus$ is deterministic because of the focalization property.

Moreover we should point out that this version of polarized $\oplus$ softness has exactly the same form as a splitting $\otimes$ in MALLP defined as follows:

Definition 7.8 (splitting a focalized $\otimes$) Under the same notation for $\uparrow \mathcal{P}$ as in Definition 7.7, every morphism of the form

$$\mathbf{1} \longrightarrow P_1 \otimes P_2, \uparrow \mathcal{P}$$

splits through the focused $\otimes$.

Remark 7.9 The above two Definitions 7.7 and 7.8 are exactly the semantical counterpart of the focalization property for positives connectives $\oplus$ and $\otimes$ in MALLP. Note that such properties are necessary for any fully complete model.

In the next subsection, we begin by studying a multiplicative version of this splitting $\otimes$. This is done by making a connection between this property and multiplicative nonpolarized full completeness of *Dinat*-**Coh**.

7.2 Full Completeness of $pDinat$-$\mathbf{HCoh}_{+,-}$ for $\mathsf{MLLP} + \mathsf{Mix_p}$

Recall, our dinats are of the form $\rho : \mathbf{1} \to F$, where F (called the *type* of ρ) is a definable multivariant functor. A dinat is called *multiplicative* if its type is defined from an MLL formula. Similarly, a polarized dinat is called *multiplicative* if its type is an MLLP formula. If ρ is of type $P_1 \otimes P_2, \Gamma$, we say the $\otimes$ in ρ is *splitting* if we can write $\rho = \rho_1 \otimes \rho_2$ where P_i is in the type of ρ_i $(i = 1, 2)$; i.e., ρ_i is of type P_i, Γ_i such that Γ is Γ_1, Γ_2.

The next Theorem makes crucial use of the depolarizing map U in Proposition 6.4 as well as Tan's full completeness theorem for $\mathsf{MLL} + \mathsf{Mix}$ [41] in the structure *Dinat*-**Coh**. We shall also need the following particular canonical morphism of $\mathbf{HCoh}_-$, which we call "polarized Mix map" (since its depolarization becomes $A \otimes B \to A \parr B$, which is an equivalent version of Mix in the nonpolarized case).

Lemma 7.10 (Polarized Mix map in $\mathbf{HCoh}_-$) *In $\mathbf{HCoh}_-$, the identity induces a map*

$$\texttt{pMix} := \ \uparrow (E \otimes F) \xrightarrow{id_{|E|\times|F|}} \uparrow E \parr \uparrow F \quad .$$

Proof. Suppose $u \in \Gamma^*(\uparrow (E \otimes F))$. This means for some $v \in \Gamma^*(E \otimes F)$ $u \supseteq v$. Since $\#v \geq 2$, we have either $v_1 \in \Gamma^*(E)$ or $v_2 \in \Gamma^*(F)$. Obviously $\forall i\ u_i \supseteq v_i$, thus either $u_1 \in \Gamma^*(\uparrow E)$ or $u_2 \in \Gamma^*(\uparrow F)$, which means $u \in \Gamma^*(\uparrow E \parr \uparrow F)$. □

Remark 7.11 (pMix and $\mathsf{Mix_p}$) In the polarized setting the morphism `pMix` is not an equivalent form of $\mathsf{Mix_p}$ defined below Definition 7.14, which is the natural adaptation of the Mix rule to MALLP; i.e., `pMix` is derived from the rule $\mathsf{Mix_p}$ in MALLP, but not vice-versa. See Remark 7.18 below for further information.

Theorem 7.12 (splitting a focalized $\otimes$ in MLLP dinats) *Let ρ be a multiplicative $pDinat$-$\mathbf{HCoh}_{+,-}$ of the form*

$$\rho : \mathbf{1} \longrightarrow P_1 \otimes P_2, \uparrow Q_1, \ldots, \uparrow Q_n, \mathcal{X}$$

where P_i $(i = 1, 2)$ and Q_j $(j = 1, \ldots, n)$ are positive and $\mathcal{X}$ are variables. Then every such ρ splits via the focalized $P_1 \otimes P_2$.

Proof. By induction on the number of connectives in $\uparrow Q_1, \ldots \uparrow Q_n$ and by using Tan's $\mathsf{MLL} + \mathsf{Mix}$ full completeness of *Dinat*-**Coh**. In the proof, U denotes the depolarizing map of Proposition 6.4.

(Case 0). Here $n = 0$. Hence $\rho : \mathbf{1} \longrightarrow P_1 \otimes P_2, \mathcal{X}$. Then $U(\rho) : \mathbf{1} \longrightarrow U(P_1) \otimes U(P_2), \mathcal{X}$ is a multiplicative **Coh** dinat with an outermost tensor. Note that $U(\rho)$ is a denotation of a

$\mathsf{MLL}+\mathsf{Mix}$ proof by $\mathsf{MLL}+\mathsf{Mix}$ full completeness of *Dinat*-**Coh**. Recall the sequenatiality of MLL. This says that if in a sequent Γ of a proof, if every outermost connective is a tensor, then one of them can be split. This is still valid in the presence of Mix for $\mathsf{MLL}+\mathsf{Mix}$. In the case $U(\rho)$ considered here (which is a denotation of a proof), there is only one outermost connective. So by $\mathsf{MLL}+\mathsf{Mix}$ sequentiality, $U(\rho)$ factors through this tensor. Thus by virtue of the type of ρ, it is direct to observe that the original ρ factors through the tensor.

(Case 1) The case where some Q_i is of the form $\downarrow N$. In this case, $\uparrow Q_i = \uparrow\downarrow N$. By composing with the co-unit $\varepsilon : \uparrow\downarrow N \longrightarrow N$ of the adjunction (7), we have

$$\varepsilon \circ \rho : \mathbf{1} \longrightarrow P_1 \otimes P_2, \uparrow Q_1, \ldots, \uparrow Q_{i-1}, N, \uparrow Q_{i-1}, \ldots, \uparrow Q_n, \mathcal{X}.$$

Note that ε here is the identity in $\mathbf{HCoh}_{+,-}$. Also, a natural transformation of the co-unit composes with a dinat to obtain a dinat. Thus $\varepsilon \circ \rho$ is a dinat and its type has a smaller number of connectives than that of ρ. Thus, by induction hypothesis, $\varepsilon \circ \rho = \rho$ splits, since ε is the identity.

(Case 2) Negation of Case 1: i.e., for all i, $Q_i = Q_{i1} \otimes Q_{i2}$. Hence $\uparrow Q_i = \uparrow (Q_{i1} \otimes Q_{i2})$ to have

$$\mathbf{1} \xrightarrow{\rho} P_1 \otimes P_2, \uparrow (Q_{11} \otimes Q_{12}), \ldots, \uparrow (Q_{n1} \otimes Q_{n2}), \mathcal{X}.$$

Applying the depolarizing map U (of above Proposition 6.4) we have

$$\mathbf{1} \xrightarrow{U(\rho)} U(P_1) \otimes U(P_2), U(Q_{11}) \otimes U(Q_{12}), \ldots, U(Q_{n1}) \otimes U(Q_{n2}), \mathcal{X}.$$

So by $\mathsf{MLL}+\mathsf{Mix}$ full completeness in *Dinat*-**Coh**, we know one of the tensors can be split:

(Case 2.1) The case where the splitting $\otimes$ of $U(\rho)$ is the first one. In this case the corresponding $\otimes$ in ρ also splits by virtue of the type of $U(\rho)$ since U acts (by restriction) as the identity on ρ. Thus we are done.

(Case 2.2) Negation of Case 2.1: Suppose without loss of generality, the splitting tensor is $U(Q_{11}) \otimes U(Q_{12})$. Then we write $U(\rho) = \tau_1 \otimes \tau_2$. Note that $U(\rho) = \rho$ since U acts as the identity on ρ (but now considered as a family of **Coh** morphisms.) We consider the canonical "polarized Mix map" $\mathtt{pMix} : \uparrow (Q_{11} \otimes Q_{12}) \rightarrow \uparrow Q_{11} \parr \uparrow Q_{12}$ in Lemma 7.10, which is given by the identity in $\mathbf{HCoh}_-$ and determines a natural transformation. After composing with this natural transformation, we have

$$\mathtt{pMix} \circ \rho : \mathbf{1} \longrightarrow P_1 \otimes P_2, \uparrow Q_{11}, \uparrow Q_{12}, \uparrow Q_2, \ldots, \uparrow Q_n, \mathcal{X}$$

considered as a dinatural family in $\mathbf{HCoh}_-$. It is important to remark that as a family of maps, $\mathtt{pMix} \circ \rho = \rho$; however in what follows its *type* will change.

Applying the depolarization map U to $\mathtt{pMix} \circ \rho$, we obtain $U(\mathtt{pMix} \circ \rho)$ in $\mathbf{HCoh}_-$. But since the family $U(\rho) = U(\mathtt{pMix} \circ \rho)$ but with different types, we know we can write $U(\mathtt{pMix} \circ \rho) = \tau_1 \parr \tau_2$ since the original splitting tensor $U(Q_{11}) \otimes U(Q_{12})$ has (under the $\mathtt{pMix}$ map) become a $\parr$. Moreover since U is the identity on dinaturals, we have that $\mathtt{pMix} \circ \rho = \tau_1 \parr \tau_2$. Here without loss of generality, we may suppose $P_{11} \otimes P_{12}$ is contained in τ_1, say

$$\tau_1 : \mathbf{1} \rightarrow P_{11} \otimes P_{12}, \uparrow Q_{11}, \uparrow \mathcal{R}_1, \qquad \tau_2 : \mathbf{1} \rightarrow \uparrow Q_{12}, \uparrow \mathcal{R}_2,$$

where the disjoint union of $\uparrow \mathcal{R}_1$ and $\uparrow \mathcal{R}_2$ equals $\uparrow Q_2, \ldots, \uparrow Q_n, \mathcal{X}$. Now we apply the induction hypothesis to τ_1 to split the outer tensor; say $\tau_1 = \tau_{11} \otimes \tau_{12}$. Without loss of generality, we may assume $\uparrow Q_{11}$ is contained in τ_{11}, say $\tau_{11} : \mathbf{1} \to P_{11}, \uparrow Q_{11}, \uparrow \mathcal{R}_{11}$, hence $\tau_{12} : \mathbf{1} \to P_{12}, \uparrow \mathcal{R}_{12}$, where the disjoint union of $\uparrow \mathcal{R}_{11}$ and $\uparrow \mathcal{R}_{12}$ equals $\uparrow \mathcal{R}_1$. Thus $\uparrow Q_{11}$ and $\uparrow Q_{12}$ occur together in the target of $\tau_{11} \parr \tau_2 : \mathbf{1} \to P_{11}, \uparrow Q_{11} \parr \uparrow Q_{12}, \uparrow \mathcal{R}_{11}, \uparrow \mathcal{R}_2$. Hence we may write $\texttt{pMix} \circ \rho = (\tau_{11} \parr \tau_2) \otimes \tau_{12}$. Finally, since $\texttt{pMix}$ is the identity (see Lemma 7.10), we observe by virtue of the type $\tau_{11} \parr \tau_2$ that the original ρ also splits on the same $P_{11} \otimes P_{12}$. □

The following is another theorem which makes crucial use of $\mathsf{MLL} + \mathsf{Mix}$ full completeness of *Dinat*-**Coh**. This theorem and the above Theorem 7.12 will lead us to a proof of multiplicative full completeness of *pDinat*-$\mathbf{HCoh}_{+,-}$ (Theorem 7.15 below). The proof is very similar to the previous theorem (although independent of it).

Theorem 7.13 (Polarized Softness of *pDinat*-$\mathbf{HCoh}_{+,-}$ in MLLP dinats) *In* MLLP*, pDinat-*$\mathbf{HCoh}_{+,-}$ *is polarized n-soft for all natural numbers n; i.e., every multiplicative polarized dinat* $\rho : \mathbf{1} \longrightarrow \uparrow P_1 \parr \cdots \parr \uparrow P_i \parr \cdots \parr \uparrow P_n \parr \mathcal{X}$ *in* $\mathbf{HCoh}_{+,-}$ *factors through some unit* $\eta_i : P_i \longrightarrow \uparrow P_i$ *of the adjunction (2) as follows to yield* $\rho = \eta_i \circ \rho'$*:*

$$\begin{tikzcd}
& \uparrow P_1, \ldots, P_i, \ldots, \uparrow P_n, \mathcal{X} \arrow[d, "\eta_i"] \\
\mathbf{1} \arrow[ur, dotted, "\rho'"] \arrow[r, "\rho"] & \uparrow P_1, \ldots, \uparrow P_i, \ldots, \uparrow P_n, \mathcal{X}
\end{tikzcd}$$

Proof. By induction on the length of ρ's type:
(Base Case) Since $U(\rho)$ is a denotation of an $\mathsf{MLL} + \mathsf{Mix}$ proof (cf. Example 6.5), this is the case where ρ is of the following form:

$$\rho : \mathbf{1} \longrightarrow \uparrow X^{\perp}, \uparrow \downarrow X$$

where X is a variable. The proof is a special case of the following (Case 1) of the Induction Case in which we do not require the (I.H.).

(Induction Case)
(Case 1) [6] : The case where some P_i is $\downarrow N$, hence $\uparrow P_i$ is $\uparrow \downarrow N$. By composing ρ with the co-unit $\varepsilon : \uparrow \downarrow N \longrightarrow N$ of the adjunction (7), we have

$$\varepsilon \circ \rho : \mathbf{1} \longrightarrow \uparrow P_1, \ldots, N, \ldots, \uparrow P_n, \mathcal{X}.$$

Then from the adjunction (1), we have

$$(\varepsilon \circ \rho)' : \mathbf{1} \longrightarrow \uparrow P_1, \ldots, \downarrow N, \ldots, \uparrow P_n, \mathcal{X}.$$

[6]The argument for this case depends on the fact that the adjunctions (1) and (2), hence (7) of $\mathbf{HCoh}_{+,-}$ are given by $=$.

In this case it is crucial to observe that in $pDinat$-$\mathbf{HCoh}_{+,-}$ the co-unit ε is $Id_{|\mathbb{N}|}$ and the adjunction for (1) is the identity; i.e., $(\varepsilon \circ \rho)' = \varepsilon \circ \rho$. Thus we have the following equation on morphisms:

$$\rho = \eta_i \circ (\varepsilon \circ \rho)'$$

This concludes the assertion.

(Case 2): Negation of (Case 1): In this case the outermost connective of P_i is $\otimes$: i.e., every P_i is of the form $P_{i1} \otimes_i P_{i2}$; thus,

$$\rho: \quad \mathbf{1} \longrightarrow \quad \uparrow(P_{11} \otimes_1 P_{12}), \ldots, \uparrow(P_{i1} \otimes_i P_{i2}), \ldots, \uparrow(P_{n1} \otimes_n P_{n2}), \mathcal{X}.$$

By applying the forgetful functor U to the dinat ρ, we have $U(\rho) \in Dinat$-$\mathbf{Coh}$

$$U(\rho): \quad \mathbf{1} \longrightarrow \quad U(P_{11}) \otimes_1 U(P_{12}), \ldots, U(P_{i1}) \otimes_i U(P_{i2}), \ldots, U(P_{n1}) \otimes_n U(P_{n2}), \mathcal{X}$$

The multiplicative full completeness of $Dinat$-$\mathbf{Coh}$ means that $U(\rho)$ is a denotation of a $\mathsf{MLL} + \mathsf{Mix}$ proof. Hence there is a splitting $\otimes$-connective for the $U(\rho)$; i.e., for some i, the tensor $\otimes_i$ splits in $U(\rho)$. Without loss of generality, we may assume that $\otimes_1$ is a splitting connective; i.e.,

$$U(\rho): \quad \mathbf{1} \longrightarrow \quad U(P_{11}) \otimes U(P_{12}), \ldots, U(\uparrow P_i), \ldots, U(\uparrow P_n), \mathcal{X}$$

factors through the first $\otimes$.

First of all, to the original ρ by applying $\mathtt{pMix} : \uparrow(P_{11} \otimes P_{12}) \to \uparrow P_{11} \invamp \uparrow P_{12}$ of Lemma 7.10, we have the following:

$$\mathtt{pMix} \circ \rho: \quad \mathbf{1} \longrightarrow \quad \uparrow P_{11} \invamp \uparrow P_{12}, \uparrow P_2, \ldots, \uparrow P_n, \mathcal{X} \tag{12}$$

Since $U(\rho)$ factors through the first $\otimes$, so does ρ of (12) via the first $\invamp$ to obtain the following ρ_1 and ρ_2 such that $\rho = \rho_1 \invamp \rho_2$:

$$\begin{array}{ll} \rho_1: \quad \mathbf{1} \longrightarrow \quad \uparrow P_{11}, \uparrow P_{i^1_1}, \ldots, \uparrow P_{i^1_{n_1}}, \mathcal{X}_1 & \\ \rho_2: \quad \mathbf{1} \longrightarrow \quad \uparrow P_{12}, \uparrow P_{i^2_1}, \ldots, \uparrow P_{i^2_{n_2}}, \mathcal{X}_2 & \text{where } \mathcal{X}_1, \mathcal{X}_2 \text{ is } \mathcal{X} \end{array}$$

The I.H.'s for ρ_1 and for ρ_2 state that $\forall k \in \{1, 2\}$, ρ_k factors either through $P_{i^k_m}$ for some m or through P_{1k}. We divide into the following two cases:

(Case 2.1) The case where *both* ρ_k's factor through P_{1k}:
After these factorizations we obtain

$$\begin{array}{l} \rho_1': \quad \mathbf{1} \longrightarrow \quad P_{11}, \uparrow P_{i^1_1}, \ldots, \uparrow P_{i^1_{n_1}}, \mathcal{X}_1 \\ \rho_2': \quad \mathbf{1} \longrightarrow \quad P_{12}, \uparrow P_{i^1_2}, \ldots, \uparrow P_{i^2_{n_2}}, \mathcal{X}_2 \end{array}$$

Then we have by applying $\otimes$

$$\rho_1' \otimes \rho_2': \quad \mathbf{1} \longrightarrow \quad P_{11} \otimes P_{12}, \uparrow P_2, \ldots, \uparrow P_n, \mathcal{X}$$

This means that ρ factors through $P_1 = P_{11} \otimes P_{12}$.

(Case 2.2) The case where *some* ρ_k factors through $P_{i^k_m}$:
After the factorization of ρ_k, we obtain

$$\rho'_k: \quad \mathbf{1} \longrightarrow \quad \uparrow P_{1k}, \ldots, P_{i^k_m}, \ldots, \uparrow P_{i^k_{n_k}}, \mathcal{X}_k$$

By making a $\parr$ with ρ_{k+1} w.r.t $\uparrow P_{11}$ and $\uparrow P_{12}$, where $k+1$ is used mod 2, we have

$$\rho'_k \parr \rho_{k+1}: \quad \mathbf{1} \longrightarrow \quad \uparrow P_{11} \parr \uparrow P_{12}, \uparrow P_2, \ldots, P_{i^k_m}, \ldots, \uparrow P_n, \mathcal{X}$$

This means that ρ of (12) factors through the $P_{i^k_m}$. Then we conclude that ρ of the assertion factors through $P_{i^k_m}$ since $\rho = \mathtt{pMix} \circ \rho$. $\square$

The following full completeness theorem is the main theorem of this subsection. In order to state it, we will need a polarized version of the Mix rule. In the next section, by applying double gluing, we prove full completeness for pure MLLP by eliminating $\mathsf{Mix_p}$.

Definition 7.14 ($\mathsf{Mix_p}$**-rule)**

$$\frac{\vdash \Gamma_1 \quad \vdash \Gamma_2}{\vdash \Gamma_1, \Gamma_2} \ \mathsf{Mix_p}$$

where at most one positive formula occurs in the conclusion of the rule.

Theorem 7.15 ($\mathsf{MLLP} + \mathsf{Mix_p}$ **full completeness of** *pDinat*-$\mathbf{HCoh}_{+,-}$)
pDinat-$\mathbf{HCoh}_{+,-}$ *is fully complete for* $\mathsf{MLLP} + \mathsf{Mix_p}$*, i.e., every polarized dinat in* *pDinat*-$\mathbf{HCoh}_{+,-}$ *is the denotation of an* $\mathsf{MLLP} + \mathsf{Mix_p}$ *proof.*

Proof. The proof is by the method of proof search to find a last rule, by induction on the type of ρ. Before beginning the proof, we first note that the polarized dinats considered in this paper are assumed to be between definable functors and that the types of dinats are legal patterns (cf. Remark 3.2). Thus in the following cases, each type of ρ is legal, which determines whether ρ is a family of morphisms in $\mathcal{C}_-$ (or dually in $\mathcal{C}_+$) or in a module $\widehat{\mathcal{C}}$.

(Base Case): The type of ρ is an axiom, say $\rho : \mathbf{1} \to X^\perp, X$. Since this ρ is a dinat of $\mathbf{Coh}$, the result is immediate from Tan's full completeness theorem for $\mathsf{MLL} + \mathsf{Mix}$.

(Case 1): The case where ρ is a family of bimodule elements; i.e., $\rho : \mathbf{1} \longrightarrow N_1, N_2, \ldots, N_k$ in *pDinat*-$\mathbf{HCoh}_{+,-}$. In this case, we may assume $N_i = \uparrow P_i$, since if not, we may replace any outermost $\parr$'s by commas. Then from polarized softness of Theorem 7.13, we know we can factor ρ through some P_i to obtain some dinat ρ'; then use the I.H.

(Case 2): The case where ρ is a family of $\mathcal{C}_-$-maps; i.e., $\rho : P^\perp \longrightarrow N_1, N_2, \ldots, N_k$ in *pDinat*-$\mathbf{HCoh}_{+,-}$.

(Case 2.1): P is $\downarrow N$. In this case, $\downarrow$ is always removable, since if $\rho : \uparrow N^\perp \to \mathcal{N}$, we may precompose with the unit $\eta : N^\perp \to \uparrow N^\perp$ of (5) to obtain a dinat family of type $N^\perp \to \mathcal{N}$, and then use the I.H.

(Case 2.2): P is $P_1 \otimes P_2$, so ρ has type $\mathbf{1} \longrightarrow P_1 \otimes P_2, \mathcal{N}$. Then by Theorem 7.12 on splitting a focalized $\otimes$, we can split ρ and apply the I.H.

(Case 2.3): P is $X^\perp$ with X a variable, so ρ has type $\mathbf{1} \longrightarrow X^\perp, \mathcal{N}$. Without loss of generality we may assume that $\mathcal{N}$ does not have any outermost $\parr$. By virtue of the fact that $U(\rho)$ of Proposition 6.4 is a denotation of an $\mathsf{MLL}+\mathsf{Mix}$ proof (see Example 6.5 (1)), $\mathcal{N}$ has exactly one occurrence of X. We instantiate every variable Y except X in $\mathcal{N}$ by $\perp$. Since $\downarrow\perp=\perp=\mathbf{1}=\uparrow\mathbf{1}$, the instantiation gives rise to a dinat $\hat{\rho}:\mathbf{1}\to X^{\perp},(\uparrow\downarrow)^m X$, where $m\geq 0$. Note that $(\uparrow\downarrow)^m$ is a sequence of polarity changing connectives which binds X in $\mathcal{N}$. But Remark 5.14 yields that $m=0$, which tells us that $\hat{\rho}:\mathbf{1}\to X^{\perp},X$. Hence this implies that $\rho:\mathbf{1}\longrightarrow X^{\perp},X,\mathcal{N}_0$, where $X,\mathcal{N}_0$ is $\mathcal{N}$. Then by using a rule $\mathsf{Mix_p}$, ρ factors into two dinats $\mathbf{1}\longrightarrow X^{\perp},X$ and $\rho':\mathbf{1}\longrightarrow\mathcal{N}_0$. □

This $\mathsf{MLLP}+\mathsf{Mix_p}$ full completeness implies the following corollary, which will be used in the proof of Main Theorem 7.21 in the next subsection:

Corollary 7.16 (From dinats to MLLP proof-structures) *Every multiplicative pDinat-$\mathbf{HCoh}_{+,-}$ is associated with a Laurent* MLLP *proof-structure.*

Proof. Every $\mathsf{MLLP}+\mathsf{Mix_p}$ proof is directly shown to be associated with an MLLP proof-structure of Laurent by induction on the length of a proof. In particular, the $\mathsf{Mix_p}$-rule corresponds to taking a disjoint union of two proof-structures for the two premises of the rule. □

The $\mathbf{HCoh}_{+,-}$ model is very special, in that although it is nondegenerate, it is also not faithful (i.e. it identifies the two proofs in Example 2.2; see also Remark 4.4). Hence, we wish to point out that full completeness implies a Läuchli-style *weakly full* completeness theorem, which is an important property of models, independently of questions of faithfulness:

Corollary 7.17 (w.f.c. of *pDinat*-$\mathbf{HCoh}_{+,-}$) *Given a sequent $A\vdash B$ in a legal pattern, if pDinat-$\mathbf{HCoh}_{+,-}(\llbracket A\rrbracket,\llbracket B\rrbracket)\neq\emptyset$, then $A\vdash B$ is provable in* $\mathsf{MLLP}+\mathsf{Mix_p}$.

7.3 The Main Theorem: Full Completeness of *pDinat*-$\mathbf{GHCoh}_{+,-}$ for MLLP

In this section we extend Theorem 7.15 to obtain our main theorem: the full completeness of *pDinat*-$\mathbf{GHCoh}_{+,-}$ for the theory MLLP of polarized MLL. The main idea is to use double gluing to kill-off the $\mathsf{Mix_p}$ rule. We assume the reader has read Subsection 5.3 on polarized double gluing categories.

7.3.1 Elimination of Polarized Mix

A crucial property of the double gluing construction $\mathbf{G}\mathcal{C}$ is that while many properties of the underlying category $\mathcal{C}$ are preserved (e.g. being a MALL category), some unwanted morphisms of the base category $\mathcal{C}$ are killed off. A typical example of such a morphism is the Mix map of $\mathcal{C}$. In our previous work [10], double gluing over $\mathbf{HCoh}$ is used mainly to kill the Mix rule, in order to obtain a pure MALL category. We remark that this situation will also hold in the polarized case under an appropriate adaptation of the Mix rule.

Remark 7.18 ($\mathbf{G}\mathcal{C}_{+,-}$ and $\mathsf{Mix_p}$) $\mathbf{G}\mathcal{C}_{+,-}$ does not necessarily support the $\mathsf{Mix_p}$-rule of Definition 7.14 even if $\mathcal{C}_{+,-}$ does:

As an example of the above remark, let us consider a MLLP provable sequent $\vdash\uparrow\mathbf{1}_G$ as premises of $\mathsf{Mix_p}$, where $\mathbf{1}_G$ denotes the tensor unit for $\mathbf{G}\mathcal{C}_{+,-}$ given by $\mathbf{1}_G=(\mathbf{1},\{id_{\mathbf{1}}\},\mathcal{C}(\mathbf{1},\mathbf{1}))$. Then the interpretation $\llbracket\vdash\uparrow\mathbf{1}_G,\uparrow\mathbf{1}_G\rrbracket$ of the lower sequent of $\mathsf{Mix_p}$ is given by the following module

$$\mathbf{1}_G \multimapdotboth\ \uparrow\mathbf{1}_G\ \invamp\uparrow\mathbf{1}_G=(\uparrow\mathbf{1}\ \invamp\uparrow\mathbf{1},\ \mathbf{G}\mathcal{C}(\downarrow\perp_G,\uparrow\mathbf{1}_G),\ \mathcal{C}(\mathbf{1},\mathbf{1})\otimes\mathcal{C}(\mathbf{1},\mathbf{1}))$$

If this module is empty, then $\mathsf{Mix_p}$ is not valid for these premises. Consider a sufficient condition for emptiness: first, if the set of copoints $\mathbf{G}\mathcal{C}(\downarrow \perp_G, \uparrow \mathbf{1}_G)$ of the above target is empty, then the module is empty. Second, given that $(\downarrow \perp_G)_p = \mathcal{C}(\mathbf{1}, \mathbf{1})$ and $(\uparrow \mathbf{1}_G)_p = \{id_{\mathbf{1}}\}$, the copoint is empty if the following holds:

$$\mathcal{C}(\mathbf{1}, \mathbf{1}) \neq \{id_{\mathbf{1}}\} \tag{13}$$

In the non-polarized case the condition (13) is known to be also a necessary condition for a double gluing category to not support Mix. This is known from one of the equivalent forms of Mix, for example, one of $A \otimes B \vdash A \parr B$ or $\perp \vdash \mathbf{1}$. However in the polarized case, both of these equivalent forms violate polarized restrictions on provable sequents.

7.3.2 The Main Theorem: MLLP Full Completeness of $pDinat$-$\mathbf{GHCoh}_{+,-}$

In what follows, we concentrate our attention on $\mathbf{G}\mathcal{C}_{+,-}$ with $\mathcal{C}_{+,-} = \mathbf{HCoh}_{+,-}$. First note that $\mathbf{GHCoh}_{+,-}$ does not support $\mathsf{Mix_p}$ since it satisfies the sufficient condition (13). Starting from this, we shall refine the previous full completeness of Theorem 7.15 for $\mathbf{HCoh}_{+,-}$ to $\mathbf{GHCoh}_{+,-}$.

Second, we note that the canonical depolarization U of Proposition 6.4 is lifted via double gluing; i.e.,

Proposition 7.19 (From $\mathbf{GHCoh}_{+,-}$ to GCoh) *There is always a canonical "depolarizing" map U which acts as an identity map such that $U(\rho) = \rho$:*

$$U : \ pDinat\text{-}\mathbf{GHCoh}_{+,-} \longrightarrow Dinat\text{-}\mathbf{GCoh}$$

Then we remark

Remark 7.20 $Dinat$-**GCoh** is fully complete for MLL (without Mix) (see Proposition 3.17 of Blute-Hamano-Scott [10]). Hence in Proposition 7.19 if a dinat ρ is definable from MLLP formula, its depolarization $U(\rho)$ is a denotation of an MLL proof.

The following full completeness theorem is the main theorem of this paper. It refines Theorem 7.15 of the previous subsection by eliminating the $\mathsf{Mix_p}$ rule:

Theorem 7.21 (MLLP full completeness of $pDinat$-$\mathbf{GHCoh}_{+,-}$)
$pDinat$-$\mathbf{GHCoh}_{+,-}$ is fully complete for MLLP*, i.e. every polarized dinat in $pDinat$-$\mathbf{GHCoh}_{+,-}$ is the denotation of an* MLLP *proof.*

Proof. We shall prove that every polarized dinat ρ is a denotation of an MLLP proof by double induction on (m_ρ, n_ρ) where m_ρ is the number of $\downarrow$'s and $\uparrow$'s in the type of ρ and n_ρ is the number of connectives outside any scopes of $\downarrow$ in the type of ρ. In the proof U denotes the canonical depolarization of Proposition 7.19. Note that as in Theorem 7.15, the polarized dinats are assumed to be between definable functors with legal patterns.
(Base Case) This is the case where the type of ρ is MLL. Since $U(\rho)$ is ρ in this case, the theorem is by MLL full completeness of $Dinat$-**GCoh** (see Remark 7.20).

For the induction case below, we begin by observing the following:
Every polarized dinat ρ in $pDinat$-$\mathbf{GHCoh}_{+,-}$ is also in $pDinat$-$\mathbf{HCoh}_{+,-}$ via the canonical

forgetful functor on double gluing. Thus Proposition 7.12 holds for ρ. Moreover from Corollary 7.16, every ρ is associated with an MLLP proof-structure Θ_ρ. [7] Let $U(\Theta_\rho)$ denote the resulting MLL proof-structure from a MLLP p-s Θ_ρ by forgetting $\{\uparrow, \downarrow\}$-links together with $\downarrow$-boxes. Then $U(\Theta_\rho)$ coincides with the MLL p-s $\Theta_{U(\rho)}$ associated with **GCoh**-dinat $U(\rho)$. In particular $\Theta_{U(\rho)}$ is connected since $U(\rho)$ is a denotation of an MLL proof.

(Induction case) Let $\Theta_1, \dots, \Theta_n$ be outermost $\downarrow$-boxes or axioms in Θ_ρ so that Θ_ρ is obtained from a union of $\Theta_1, \dots, \Theta_n$ by drawing $\{\parr, \otimes, \uparrow\}$-links hereditarily below some Θ_i: If there is no such link hereditarily below, then since $\Theta_{U(\rho)}$, which is $U(\Theta_\rho)$, is connected, n must be 1 and Θ_1 (which is Θ_ρ itself) is a $\downarrow$-box. In this case we can always eliminate the principal $\downarrow$-link to obtain a dinat of a smaller size. Hence in what follows, we assume there must exist at least one such link hereditarily below some Θ_i (outside of all $\Theta_1, \dots, \Theta_n$). The proof goes through the following three steps:

(Step 1) If there is a bottom-most $\parr$-link, we can eliminate it to obtain a dinat of smaller size.

(Step 2) If there is a bottom-most $\otimes$-link after Step 1, Proposition 7.12 says that the $\otimes$ splits to obtain two dinats of smaller sizes.

(Step 3) After Step 1 and 2, all bottom-most links hereditarily below $\Theta_1, \dots, \Theta_n$ are $\uparrow$. Since these $\uparrow$-links are outside $\Theta_1, \dots, \Theta_n$, any of them can be removed to obtain a dinat of a smaller size. $\square$

Finally, we wish to point out that full completeness implies a Läuchli-style *weakly full* completeness theorem, which is an important property of models independently of faithfulness:

Corollary 7.22 (w.f.c. of ***pDinat*****-GHCoh$_{+,-}$)** *If pDinat-***GHCoh**$_{+,-}(\llbracket A \rrbracket, \llbracket B \rrbracket) \neq \emptyset$, *then* $A \vdash B$ *is provable in* MLLP.

7.4 Polarized Gustave functions in HCoh$_{+,-}$ and the failure of full completeness for MALLP

This section proves a curious property of **HCoh**$_{+,-}$, in contrast to ordinary hypercoherences **HCoh**, which directly implies that our full completeness theorems Theorem 7.15 and Theorem 7.21 for polarized dinaturals cannot be extended to include additives; i.e., our results cannot be extended to larger fragments of MALLP than just MLLP.

Ehrhard's category **HCoh** is a refinement of Girard's original **Coh**. One of the most important reasons for this is that **HCoh** kills Gustave functions in **Coh** (See [10]). Gustave functions are analogs of parallel-or in domain theory (see [3]) and are intimately related to studies of sequentiality [18]. But contrary to this phenomenon in usual linear logic, in the polarized hypercoherences defined in previous subsections, a polarized version of a Gustave function turns out to be a morphism of **HCoh**$_{+,-}$. This directly implies that **HCoh**$_{+,-}$ is not fully complete for MALLP (cf. Corollary 7.27).

Let us start this subsection by considering the functor $\downarrow\uparrow$:

Proposition 7.23 *The functor* $\downarrow\uparrow$ *:***HCoh**$\longrightarrow$ **HCoh**$_+$ *satisfies:*

[7] Our use of proof-structures is not essential in that we do not use O. Laurent's correctness criterion. We also have an alternative direct proof without associating p-s's, similar to the proof of Theorem 7.15.

- $\downarrow\uparrow$ *preserves linear connectives* $\otimes, \parr, \oplus, \&$.
- $\downarrow\uparrow$ *forgets the polarity change connectives* $\downarrow$ *and* $\uparrow$*: i.e., for all* $E \in \mathbf{HCoh}$,

$$\begin{aligned} \downarrow\uparrow(\downarrow E) &= \downarrow\uparrow(E) && (14)\\ \downarrow\uparrow(\uparrow E) &= \downarrow\uparrow(E) && (15) \end{aligned}$$

Proof. The preservation of linear connectives is directly checked by using Proposition 5.15, in particular the characterization (11) for u to belong to $\Gamma(\downarrow\uparrow E)$. Thus we go to:

Proof of (14):
($\subseteq$) Obvious by applying the functor $\downarrow\uparrow$ to the counit $\downarrow E \longrightarrow E$ of the adjunction (1).
($\supseteq$) Let us apply the adjunction (1) to the assertion:

$$\frac{\downarrow\uparrow(E) \longrightarrow \downarrow\uparrow(\downarrow E)}{\downarrow\uparrow(E) \longrightarrow \uparrow(\downarrow E)}\ \mathsf{Inj}_+ \dashv \downarrow$$

Thus the assertion is equivalent to

$$\Gamma^*(\downarrow\uparrow E) \quad\subseteq\quad \Gamma^*(\uparrow\downarrow E)$$

Let us calculate (where we write $\subseteq_{\geq 2}$ to mean "subset of cardinality ≥ 2").

$$\begin{aligned} u \in \Gamma^*(\downarrow\uparrow E) &\Leftrightarrow \forall u' \subseteq_{\geq 2} u\ u' \in \Gamma^*(\uparrow E)\\ &\Leftrightarrow \forall u' \subseteq_{\geq 2} u\ \exists u'' \subseteq u'\ u'' \in \Gamma^*(E) \end{aligned}$$

$$\begin{aligned} v \in \Gamma^*(\uparrow\downarrow E) &\Leftrightarrow \exists v' \subseteq_{\geq 2} v\ v' \in \Gamma^*(\downarrow E)\\ &\Leftrightarrow \exists v' \subseteq_{\geq 2} v\ \forall v'' \subseteq_{\geq 2} v'\ v'' \in \Gamma^*(E) \end{aligned}$$

From this $u \in \Gamma^*(\downarrow\uparrow E)$ implies that $\forall u' \subseteq u(\#u = 2 \Rightarrow u' \in \Gamma^*(E))$. But this implies $u \in \Gamma^*(\uparrow\downarrow E)$ by taking (in the above expression) v, v' to be u itself. *End of Proof of (14)*

Proof of (15): This is obvious since $\uparrow\uparrow = \uparrow$, which implies the assertion by applying the functor $\downarrow$. *End of Proof of (15)*

□

The following lemma is crucial in showing that Polarized Gustave is in $\mathbf{HCoh}_{+,-}$.

Lemma 7.24 *For every hypercoherence* $E \in \mathbf{HCoh}$,

$$\mathbf{Coh}(\mathbf{1}, \downarrow\uparrow E) \quad\subseteq\quad \mathbf{HCoh}(\mathbf{1}, \uparrow E)$$

Proof. From Proposition 5.15 and of the following natural transformation

$$\downarrow\uparrow \quad\longrightarrow\quad \uparrow$$

which is derived from the counit (4) of the adjunction (1)

$$\downarrow \quad\longrightarrow\quad id$$

by applying the functor $\uparrow$. □

Proposition 7.25 (3-ary Polarized Gustave in $\mathbf{HCoh}_{+,-}$) *The following is a morphism in* $\mathbf{HCoh}_{+,-}$

$$R : \mathbf{1} \quad \longrightarrow \quad \uparrow(\downarrow(X\&Y) \oplus Z),\ \uparrow(\downarrow(Y\& \uparrow Z)\oplus \downarrow X),\ \uparrow(\downarrow(\uparrow Z\&X)\oplus \downarrow Y)$$

where $Z = X^{\perp} \otimes Y^{\perp}$ *and* $X, Y \in \mathbf{HCoh}_{-}$.

$$\begin{array}{rcl} R & := & \{((1,a_1),(2,a_3),(3,a_2)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ & \cup & \\ & & \{((3,a_3),(1,a_2),(2,a_1)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ & \cup & \\ & & \{((2,a_2),(3,a_1),(1,a_3)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ & \cup & \\ & & \{((1,a_1),(1,a_2),(1,a_3)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \\ & \cup & \\ & & \{((2,a_2),(2,a_3),(2,a_1)) \mid a_1 \in |E_1| \wedge a_2 \in |E_2| \wedge a_3 = (a_1,a_2)\} \end{array}$$

Proof. First, the codomain of R is of the form

$$\uparrow E_1 \parr \uparrow E_2 \parr \uparrow E_3,$$

which is equal to the following, by Lemma 5.16:

$$\uparrow(E_1 \parr E_2 \parr E_3)$$

Second, on the other hand, by Proposition 7.23, we have the following for $X, Y, Z \in \mathbf{Coh}$;

$$\downarrow\uparrow(E_1 \parr E_2 \parr E_3) \quad = \quad ((X\&Y) \oplus Z) \parr ((Y\&Z) \oplus X) \parr ((Z\&X) \oplus Y).$$

It is known in MALL that R is a Gustave function in $\mathbf{Coh}$ (See Proposition 2.11 and the following paragraph in Blute-Hamano-Scott [10]); i.e.,

$$R \in \mathbf{Coh}(\mathbf{1}, ((X\&Y) \oplus Z) \parr ((Y\&Z) \oplus X) \parr ((Z\&X) \oplus Y))$$

So the assertion follows from Lemma 7.24. $\square$

We then have

Proposition 7.26 *The Gustave R in Proposition 7.25 is not the denotation of a* MALLP *proof.*

Proof. Suppose for contradiction that R is a denotation of a MALLP proof π. From the cut-elimination theorem of MALLP and Proposition 4.2 (2), we may assume that π is cut-free. Thus the last rule of π must be $\uparrow$, hence its premise must have a focalized $\oplus$. Then, from the focalization property of MALLP (Proposition 2.4), the second last rule of π must be $\oplus_i$ with some $i \in \{1, 2\}$ in order to introduce the focalized $\oplus$. However this contradicts the fact that R contains both elements from the 1st and 2nd components of any possible focalized $\oplus$ (i.e. R cannot factor through any $\oplus_i$). $\square$

This proposition implies the following corollary:

Corollary 7.27 $\mathbf{HCoh}_{+,-}$ *is not fully complete for* MALLP.

Remark 7.28 (Extending Gustave to dinaturals and double gluing) The above Gustave R is also a morphism of the double gluing category $\mathbf{GHCoh}_{+,-}$ of Section 5.3. On the otrher hand, the above Gustave R extends to the stronger framework of functoriality of Section 6, in that $R = R_{X,Y}$ actually determines a polarized dinatural family for $\mathbf{HCoh}_{+,-}$. Moreover this is still the case for the double gluing category.

From this remark, similarly as above, we have more generally
Corollary 7.29

- *Neither* $\mathbf{HCoh}_{+,-}$ *nor pDinat-*$\mathbf{HCoh}_{+,-}$ *is fully complete for* MALLP $+\mathsf{Mix}_{\mathsf{p}}$.
- *Neither* $\mathbf{GHCoh}_{+,-}$ *nor pDinat-*$\mathbf{GHCoh}_{+,-}$ *is fully complete for* MALLP.

8 Some Remarks on Faithfulness

We say that a polarized category $\mathcal{C}_{+,-}$ is faithful when $[\![\pi_1]\!] \neq [\![\pi_2]\!]$ for "different" MALLP proofs π_1 and π_2. Of course, to make this problem precise, we need a good theory of equations between MALLP proofs. In what sense do proofs form (the morphisms of) a category? Let us consider the problem of categories of additive proof nets. These have been looked at in various works of Cockett and Seely [9, 15] as well as recent work on multiplicative unitless categories of proof nets of Hughes, Houston and Schalk [27] (and the literature cited there) aiming towards the additive case. See also the recent book of Došen and Petrić ([17], Chapter 4).

At the additive level, the advantage of MALLP to nonpolarized MALL is that O. Laurent's nets do form a category (using composition as cut) since they obey both Church-Rosser and Strong Normalization, as well as having a good theory of units. In fact they form a pre-$*$-autonomous polarized framework in our sense, although they do not form a polarized $*$-autonomous category. Even here, however, the exact notion of equations is delicate; for example, to have genuine products we must insist on the equations of surjective pairing. However, once having agreed on such equations, this permits us to precisely define the problem of faithfulness of the interpretation $[\![-]\!]$ of a syntactic proof net category into any concrete polarized model $\mathcal{C}_{+,-}$.

We are interested in faithfulness of $[\![-]\!]$ and finding conditions guaranteeing it, especially when the categories $\mathcal{C}_{+,-}$ are the concrete examples of the previous sections. Let us begin by presenting a necessary criterion for faithfulness. This criterion is concerned with descending sequences of subcategories of $\mathcal{C}_-$ and $\mathcal{C}_+$ containing $\mathsf{Fix}\epsilon$ and $\mathsf{Fix}\eta$ respectively.

Definition 8.1 (Decreasing sequence $\{\ (\downarrow\uparrow)^n\mathcal{C}\ \}_{n\geq 1}$ of subcategories containing $\mathsf{Fix}\eta$) In $\mathcal{C}_+$ (resp. $\mathcal{C}_-$), starting from $\downarrow\uparrow\mathcal{C}$ (resp. $\uparrow\downarrow\mathcal{C}$), there is a decreasing sequence of subcategories containing $\mathsf{Fix}\eta$ (resp. $\mathsf{Fix}\epsilon$):

$$\downarrow\uparrow\mathcal{C} \supseteq (\downarrow\uparrow)^2\mathcal{C} \supseteq \cdots \supseteq (\downarrow\uparrow)^n\mathcal{C} \supseteq \cdots\cdots \supseteq \mathsf{Fix}\eta$$

$$(\text{resp. } \uparrow\downarrow\mathcal{C} \supseteq (\uparrow\downarrow)^2\mathcal{C} \supseteq \cdots \supseteq (\uparrow\downarrow)^n\mathcal{C} \supseteq \cdots\cdots \supseteq \mathsf{Fix}\epsilon)$$

The above sequence is said to *terminate* if the following holds;

$$(\downarrow\uparrow)^n\mathcal{C} = (\downarrow\uparrow)^{n+1}\mathcal{C} \ \text{ for some } n. \tag{16}$$

Equivalently,

$$(\downarrow\uparrow)^n = (\downarrow\uparrow)^{n+1} \ \text{ for some } n,$$

where $(\downarrow\uparrow)^m$ is a functor $(\downarrow\uparrow)^m : \mathcal{C} \to \mathcal{C}_+$ defined inductively on m.

We should note that the sequence in Definition 8.1 does not terminate in general, because in the syntax of MALLP there is no "provable isomorphism" between the two formulas $(\downarrow\uparrow)^{n+1}P$ and $(\downarrow\uparrow)^nP$; this means no cut-free proof of an η-expansion of the identity axiom $P \vdash P$ can be obtained from the composition of any two proofs $\pi_1 : (\downarrow\uparrow)^{n+1}P \vdash (\downarrow\uparrow)^nP$ and $\pi_2 : (\downarrow\uparrow)^nP \vdash (\downarrow\uparrow)^{n+1}P$ using cut-elimination. This in particular implies that $\mathsf{Fix}\,\eta$ (resp. $\mathsf{Fix}\,\epsilon$) does not coincide with any $(\downarrow\uparrow)^n\mathcal{C}$ (resp. $(\uparrow\downarrow)^n\mathcal{C}$), hence with $\mathcal{C}_+$ (resp. $\mathcal{C}_-$)

Proposition 8.2 (A criterion for faithfulness) *A necessary condition for a categorical model $\mathcal{C}_{+,-}$ to be faithful for* MALLP *is that the above infinite sequence of operations $(\downarrow\uparrow)^n$ does not terminate, i.e. the negation of the above condition (16) holds. In particular, the criterion is sufficient to distinguish the two proofs of Example 2.2.*

Unfortunately all our concrete models in the previous sections are not faithful since they fail to satisfy this criterion. More precisely we have:

Remark 8.3 ($\uparrow\downarrow$-Sequences in our examples terminate)

- $\uparrow$ and $\downarrow$ act as the identity on morphisms of $\mathbf{HCoh}_{+,-}$, $\mathbf{GHCoh}_{+,-}$ and $(\mathbf{G}^2\mathcal{C})_{+,-}$, hence the sequence is trivial.

- $(\uparrow\downarrow)^2 = (\uparrow\downarrow)^1$ for $\mathbf{Chu}_{+,-}$, hence the sequence terminates.

This remark leads to the question of whether one can find any *syntax-free* example of a polarized (or even multiplicative fragment) category which is faithful. This question will be studied in our paper [25] in preparation. The fundamental idea of [25] is to first find a $*$-autonomous category $\mathcal{C}$ with (co-)products so as to interpret $\downarrow$ and $\uparrow$ by products/coproducts:

$$\downarrow N := \bot \,\&N \quad \text{and} \quad \uparrow P := \mathbf{1} \oplus P \tag{17}$$

Second, we then impose a certain *uniformity* condition on $\mathcal{C}$-morphisms so as to build an MLLP subcategory in $\mathcal{C}$, in particular to obtain adjunctions $\mathsf{Inj}_+ \dashv \downarrow$ and $\uparrow \dashv \mathsf{Inj}_-$ under the interpretation (17). In [25] this will be done when $\mathcal{C}$ is **HCoh** and we impose a uniformity condition from the (external use of) additive softness of **HCoh**. We note that under the interpretation of (17), there is a direct correspondence between $\oplus$-softness of MALL and $\uparrow$-softness in MLLP.

9 Conclusion, Other Results and Open Problems

We have given a general definition of polarized models for Olivier Laurent's theory MALLP, have introduced various concrete examples, and have begun investigating the problem of full completeness of such models. As a first step towards general full completeness theorems, we have restricted ourselves in this paper to the multiplicative fragment MLLP, and shown the full completeness for that fragment by using the polarized hypercoherence model.

The most important question is how to extend the above results to MALLP. On the one hand, we know *pDinat*-$\mathbf{GHCoh}_{+,-}$ cannot be fully complete for MALLP because of the existence of polarized Gustave Functions (cf. Section 7.4). However since the type of Gustave is a provable sequent for MALLP, there arises:

Question 9.1 Is *pDinat*-$\mathbf{GHCoh}_{+,-}$ weakly fully complete for MALLP ?

We can also ask which other categorical structures *pDinat*-$\mathcal{C}$ are fully or weakly fully complete for MALLP. This will be the subject of future work.

Let us also mention a fact we know about $(\mathbf{G}^2\mathcal{C})_{+,-}$ with $\mathcal{C} = \mathbf{HCoh}$:

Theorem *pDinat*-$(\mathbf{G^2HCoh})_{+,-}$ *is polarized* n*-soft for* MALLP*, for all* $n \geq 1$.

The proof is similar to the proofs of $\uparrow$-softness of *pDinat*-$\mathbf{HCoh}_{+,-}$ for MLLP in Section 7.2, but the proof is more involved in the presence of additives, as it uses the result of our paper [10] that *Dinat*-$\mathbf{GHCoh}$ is fully complete for MALL. This will appear in a future paper. Since polarized softness is a key property necessary for full completeness, the result suggests the following:

Question 9.2 Given a fully complete model for MALL (e.g. *Dinat*-$\mathbf{GHCoh}$), is there a way of constructing from it a fully complete polarized model for MALLP ?

Our polarized multiplicative full completeness in $\mathcal{C}_{+,-}$:=$\mathbf{HCoh}_{+,-}$ is carried out by reduction to non-polarized multiplicative full completeness in $\mathbf{Coh}$, which in this case happens to be a fixed point subcategory $\mathsf{Fix}\,\epsilon$ for an adjoint equivalence $\Uparrow \dashv \Downarrow$. So there arises a natural question:

Question 9.3 (reducing polarized full completeness to the non polarized case) Is there an abstract proof of a polarized full completeness theorem (for either MALLP or MLLP) for a general polarized category $\mathcal{C}_{+,-}$ by reduction to a non-polarized full completeness theorem (for MALL or MLL, resp) using the associated structure $\mathsf{Fix}\,\epsilon$?

An answer to this question may clarify how to understand abstractly our full completeness proofs for $\mathbf{HCoh}_{+,-}$ and $\mathbf{GHCoh}_{+,-}$ and hence how to generalize our polarized full completeness to a more abstract level.

Finally, the questions of faithfulness mentioned in the last section, and also their connections to various categories of polarized proof nets, is the subject of future work.

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10 Appendices

10.1 Appendix 1: On Polarized Categories à la M. Barr

M. Barr's original monograph [7] introduced $*$-autonomous categories by a slightly different route than in the recent linear logic literature (e.g. [8, 12]). Ironically, his approach is much closer to polarized linear logic and forms the basis of what we here call a *polarized category.*

The following definition is a slightly modified version of Barr's *pre-$*$-autonomous situation* [7], pp. 15-16. (see the discussion in Remark 10.6 below).

Definition 10.1 A *polarized category*, denoted $\mathcal{C}_{+,-}$, consists of the following data:

(i) A category $\mathcal{C}$ with two *full* subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$, where $\mathcal{C}_+$ is called the *positive category* and $\mathcal{C}_-$ is called the *negative category*.

(ii) There is a contravariant equivalence of the two subcategories:

$$(-)^{\perp} : (\mathcal{C}_+)^{op} \xrightarrow{\cong} \mathcal{C}_-$$

If we denote the inverse of $(\ )^{\perp}$ by $(\ )^{\#}$, the equivalence induces natural isomorphisms $P \cong P^{\perp\#}$ and $N \cong N^{\#\perp}$ for every positive object $P \in \mathcal{C}_+$ and every negative object $N \in \mathcal{C}_-$.

(iii) There is a functor $\multimap$

$$\multimap : (\mathcal{C}_+)^{op} \times \mathcal{C}_- \longrightarrow \mathcal{C}_-$$

and an object $\mathbf{1} \in \mathcal{C}_+$ satisfying:

$$\mathbf{1} \multimap N \cong N \tag{18}$$

$$\mathcal{C}(\mathbf{1}, P \multimap N) \cong \mathcal{C}(P, N) \tag{19}$$

$$P_1 \multimap (P_2 \multimap P_3^{\perp}) \cong P_3 \multimap (P_2 \multimap P_1^{\perp}) \tag{20}$$

(iv) $\mathcal{C}_-$ (resp. $\mathcal{C}_+$) form reflective (resp. coreflective) subcategories of $\mathcal{C}$ with $\uparrow$ (resp. $\downarrow$) the reflector (resp. coreflector). That is, there are functors $\uparrow : \mathcal{C} \to \mathcal{C}_-$ and $\downarrow : \mathcal{C} \to \mathcal{C}_+$ such that $\uparrow$ is left adjoint to the inclusion $\mathsf{Inj}_- : \mathcal{C}_- \hookrightarrow \mathcal{C}$ and $\downarrow$ is right adjoint to the inclusion $\mathsf{Inj}_+ : \mathcal{C}_+ \hookrightarrow \mathcal{C}$. Thus we have:

$$\mathcal{C}(\mathsf{Inj}_+(P), A) \cong \mathcal{C}_+(P, \downarrow A) \tag{21}$$

$$\mathcal{C}(A, \mathsf{Inj}_-(N)) \cong \mathcal{C}_-(\uparrow A, N) \tag{22}$$

where $P \in \mathcal{C}_+$, $A \in \mathcal{C}$ and $N \in \mathcal{C}_-$.

The units and counits of the adjunctions (21) and (22) are the same as in Definition 3.1 and so is the duality of the adjunctions.

The above (21) and (22) may be described by the following diagram:

$$\begin{tikzcd} \mathcal{C}_- \arrow[r, hook, shift right, "\mathsf{Inj}_-"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "\uparrow"'] \arrow[r, shift right, "\downarrow"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_+ \arrow[l, hook', shift right, "\mathsf{Inj}_+"'] \end{tikzcd} \tag{23}$$

Let us write $\Uparrow$ for $\uparrow \circ \mathsf{Inj}_+$ and $\Downarrow$ for $\downarrow \circ \mathsf{Inj}_-$ (composition of adjoints). Then we may write the above diagram by:

$$\begin{tikzcd} \mathcal{C}_- \arrow[r, shift right, "\Downarrow"'] \arrow[r, phantom, "\perp"] & \mathcal{C}_+ \arrow[l, shift right, "\Uparrow"'] \end{tikzcd}$$

The units and the counits of this adjunction are the same as in Definition 3.1 and so is the definability of the adjunctions.

(v) The functors $\Uparrow$ and $\Downarrow$ are DeMorgan dual to each other; i.e. there are natural isomorphisms

$$\begin{aligned} (\Downarrow N)^\perp &\cong \Uparrow (N^\#) \\ (\Uparrow P)^\# &\cong \Downarrow (P^\perp) \end{aligned}$$

for all $N \in \mathcal{C}_-$ and $P \in \mathcal{C}_+$.

Remark 10.2 We follow Barr's notation in denoting the inverse of $(-)^\perp$ by $(-)^\#$. In the case of a polarized $*$-autonomous category as in Definition 3.1 (so the ambient category $\mathcal{C}$ is $*$-autonomous) the functor $(-)^\perp : (\mathcal{C}_+)^{op} \xrightarrow{\cong} \mathcal{C}_-$ is induced by a contravariant duality $(-)^\perp$ on $\mathcal{C}$. In this case, $(-)^\# : (\mathcal{C}_-)^{op} \xrightarrow{\cong} \mathcal{C}_+$ is also induced by the same ambient $(-)^\perp$ and indeed we can identify $\# = \perp$ in this case. This includes the case of the syntax of MALLP (where we can define a strictly involutive negation on all formulas as in the remarks at the end of Section 2) as well as the case of all the examples of Section 5 above. For more details, see also Remark 10.6 below.

The functors $\multimap$ together with $(\ )^\perp$ and $(\ )^\#$ induce monoidal structures $\otimes$ and $⅋$ respectively on $\mathcal{C}_+$ and $\mathcal{C}_-$:

Proposition 10.3 (monoidal structures on $\mathcal{C}_+$ and $\mathcal{C}_-$)

- *On the positive category $\mathcal{C}_+$, the functors $\multimap$, $(\ )^\perp$, and $(\ )^\#$ induce a symmetric monoidal structure with* tensor product $\otimes$ *defined by:*

$$P \otimes Q := (P \multimap Q^\perp)^\#$$

Note that $\mathbf{1}$ *in $\mathcal{C}_+$ becomes a unit for the tensor.*

- *Dually, on the negative category $\mathcal{C}_-$, the functors $\multimap$, $(\ )^\perp$ and $(\ )^\#$ induce a symmetric monoidal structure with* cotensor $⅋$ *defined by:*

$$M ⅋ N := M^\# \multimap N$$

Note that $\perp := \mathbf{1}^\perp$ in $\mathcal{C}_-$ becomes a unit for the cotensor.

This weaker form of pre-$*$-autonomous situation in our definition of $\mathcal{C}_{+,-}$ implies the following:

Proposition 10.4 (closedness of $\otimes$ in the pattern $+ \to -$) *For objects $P, Q \in \mathcal{C}_+$ and $N \in \mathcal{C}_-$, there exists a natural bijection between hom-sets of $\mathcal{C}$ from positive to negative objects:*

$$\mathcal{C}(P \otimes Q, N) \;\cong\; \mathcal{C}(Q, P^\perp ⅋ N) \tag{24}$$

The above suffices for understanding MLLP. For MALLP, we adjoin additional structure.

Definition 10.5 (Polarized categories with Additives) Let $\mathcal{C}_{+,-}$ be a polarized category. We adjoin additives by assuming the following additional structure:

- The negative category $\mathcal{C}_-$ has products, denoted &, with unit T.
- $\mathcal{C}_-$ has a natural distributive law

$$M \parr (N \& L) \;\cong\; (M \parr N) \& (M \parr L) \tag{25}$$

- $\mathcal{C}_+$ has a natural distributive law

$$P \otimes (Q \oplus R) \;\cong\; (P \otimes Q) \oplus (P \otimes R) \tag{26}$$

  where $\oplus$ denotes the induced *coproduct* in $\mathcal{C}_+$ via the dualizing functor, given by $P_1 \oplus P_2 := (P_1^\perp \& P_2^\perp)^\#$ with its unit $0 = T^\#$.

Finally we remark that in the case when the ambient category $\mathcal{C}$ is $*$-autonomous, as in our paper above, we may take $\perp = \#$ and the distributive laws in $\mathcal{C}_-$ and $\mathcal{C}_+$ are consequences of $*$-autonomy.

Remark 10.6 (On Barr's pre-$*$-autonomous situations) Let us compare our definition of polarized category to Barr's *pre-$*$-autonomous situation* (cf. Def 4.6 of pg. 16 of [7]). Condition (i) is the same. Conditions (ii) and (iii) of the above definition of polarized category are a slightly weaker form of Barr's, which we explain below. Finally, conditions (iv) and (v) are slightly stronger, since we demand the full subcategories of negative and positive objects be reflective (resp. coreflective). This latter condition corresponds to the polarity changing operations of O. Laurent's polarized logics, and is crucial to our framework.

Re conditions (ii) and (iii), the only one of Barr's conditions missing here is $\mathcal{C}(P^\perp, Q) \cong \mathcal{C}(Q^\perp, P)$ for $P, Q \in \mathcal{C}_+$, which is not necessary in our framework since the pattern of these maps is $- \to +$, which is illegal, i.e. is not allowed (by the focussing property) in MALLP proofs (See Remark 3.2). Barr [7], pp. 15-16, extends the contravariant equivalence $(-)^\perp : (\mathcal{C}_+)^{op} \xrightarrow{\cong} \mathcal{C}_-$ to the union $\mathcal{C}_+ \cup \mathcal{C}_-$. In particular, using Barr's notation, if we denote the inverse of $(-)^\perp$ by $(-)^\# : (\mathcal{C}_-)^{op} \xrightarrow{\cong} \mathcal{C}_+$, Barr proves that $(-)^\perp$ and $(-)^\#$ coincide on $\mathcal{C}_+ \cap \mathcal{C}_-$, using various coherence conditions (see (4.7) of [7]) which use the illegal pattern above in their proof. The coincidence of $(-)^\perp$ and $(-)^\#$ on $\mathcal{C}_+ \cap \mathcal{C}_-$ is necessary for the equivalence $(-)^\perp : (\mathcal{C}_+)^{op} \xrightarrow{\cong} \mathcal{C}_-$ to extend to the union $\mathcal{C}_+ \cup \mathcal{C}_-$. In our setting, the functor $(\ )^\perp$ and its inverse $(\ )^\#$ need not coincide on the intersection of the two subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$. However, for the purposes of modelling MALLP in this paper (where the ambient category $\mathcal{C}$ is $*$-autonomous), we do not need this additional structure.

Notice this does suggest more general polarized categories, with two "negations", satisfying $P^{\perp\#} \cong P$ and $N^{\#\perp} \cong N$, along with a more general polarized logical syntax. This also suggests that some of the more concrete topological examples from Barr's monograph may serve as models of polarized logics. These issues are currently under investigation.

10.2 Appendix 2: Double Gluing

We recall the definition of double gluing from our paper [10], following Tan and Hyland-Schalk [41, 30].

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, (-)^\perp)$ be a $*$-autonomous category. Let H denote the covariant *points* functor $\mathcal{C}(\mathbf{1}, -) : \mathcal{C} \longrightarrow \mathbf{Set}$ and K denote the contravariant *copoints* functor $\mathcal{C}(-, \mathbf{1}^\perp) \;\cong\; \mathcal{C}(\mathbf{1}, (-)^\perp) : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$.

Definition 10.7 The category $\mathbf{G}\mathcal{C}$, the *double gluing category* of $\mathcal{C}$, has objects triples $\mathcal{A} = (A, \mathcal{A}_p, \mathcal{A}_{cp})$ where $A := |\mathcal{A}|$ is an object of $\mathcal{C}$, where $\mathcal{A}_p \subseteq H(|\mathcal{A}|) = \mathcal{C}(\mathbf{1}, A)$ is a set of points of A and $\mathcal{A}_{cp} \subseteq K(|\mathcal{A}|) = \mathcal{C}(A, \mathbf{1}^\perp) \cong \mathcal{C}(\mathbf{1}, A^\perp)$ is a set of copoints of A .

A morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbf{G}\mathcal{C}$ is a morphism $f : |\mathcal{A}| \longrightarrow |\mathcal{B}|$ in $\mathcal{C}$ such that $Hf : \mathcal{A}_p \longrightarrow \mathcal{B}_p$ and $Kf : \mathcal{B}_{cp} \longrightarrow \mathcal{A}_{cp}$ are well-defined **Set**-maps so that $f(\mathcal{A}_p) \subseteq \mathcal{B}_p$ and $f^\perp(\mathcal{B}_{cp}) \subseteq \mathcal{A}_{cp}$; i.e., the following conditions hold:

(point condition:) $\quad \forall \alpha \in \mathcal{A}_p \; [\alpha] f \in \mathcal{B}_p$

(copoint condition:) $\quad \forall \beta \in \mathcal{B}_{cp} \; f[\beta] \in \mathcal{A}_{cp}$

Given $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow \mathcal{C}$ in $\mathbf{G}\mathcal{C}$, the composition $gf : \mathcal{A} \longrightarrow \mathcal{C}$ is induced from the underlying composition in $\mathcal{C}$. Similarly, the identity morphism on $\mathcal{A}$ is given by the identity morphism on $|\mathcal{A}|$ in $\mathcal{C}$.

Fact 10.8 *For any $*$-autonomous category $\mathcal{C}$, $\mathbf{G}\mathcal{C}$ is a $*$-autonomous category.*

Proof. We define $(\;)^\perp$ (linear negation) by the formula:

$$\mathcal{A}^\perp = (|\mathcal{A}|^\perp, \mathcal{A}_{cp}, \mathcal{A}_p)$$

We define the tensor product $\mathcal{A} \otimes \mathcal{B}$ as follows:

$$\mathcal{A} \otimes \mathcal{B} = (|\mathcal{A}| \otimes |\mathcal{B}|, (\mathcal{A} \otimes \mathcal{B})_p, (\mathcal{A} \otimes \mathcal{B})_{cp})$$

where

$$\begin{array}{rcl} (\mathcal{A} \otimes \mathcal{B})_p & = & \{\alpha \otimes \beta | \alpha \in \mathcal{A}_p, \beta \in \mathcal{B}_p\} \\ (\mathcal{A} \otimes \mathcal{B})_{cp} & = & \mathbf{G}\mathcal{C}(\mathcal{A}, \mathcal{B}^\perp) \end{array}$$

Note that this last equality makes sense, because:

$$\mathbf{G}\mathcal{C}(\mathcal{A}, \mathcal{B}^\perp) \subseteq \mathcal{C}(|\mathcal{A}|, |\mathcal{B}|^\perp) \cong \mathcal{C}(|\mathcal{A}| \otimes |\mathcal{B}|, \mathbf{1}^\perp)$$

We also define the unit for the tensor product by $\mathbf{1_G} = (\mathbf{1}, \{id_\mathbf{1}\}, \mathcal{C}(\mathbf{1}, \mathbf{1}))$.
We thus obtain that $\mathbf{G}\mathcal{C}$ is $*$-autonomous. □

In fact, if $\mathcal{C}$ has products (thus coproducts), so does $\mathbf{G}\mathcal{C}$, as follows:

Definition 10.9 (Products and Coproducts in $\mathbf{G}\mathcal{C}$)

Product

$$\mathcal{A} \& \mathcal{B} = (|\mathcal{A}| \& |\mathcal{B}|, (\mathcal{A} \& \mathcal{B})_p, (\mathcal{A} \& \mathcal{B})_{cp})$$

where

$$\begin{array}{rcl} |\mathcal{A}| \& |\mathcal{B}| & & \text{is the product in } \mathcal{C} \\ (\mathcal{A} \& \mathcal{B})_p & = & \{\alpha + \beta \mid \alpha \in \mathcal{A}_p \text{ and } \beta \in \mathcal{B}_p\} \\ (\mathcal{A} \& \mathcal{B})_{cp} & = & \mathcal{A}_{cp} + \mathcal{B}_{cp} \quad \text{where } + \text{ denotes the disjoint union} \end{array}$$

Coproduct

$$\mathcal{A} \oplus \mathcal{B} = (|\mathcal{A}| \oplus |\mathcal{B}|, (\mathcal{A} \oplus \mathcal{B})_p, (\mathcal{A} \oplus \mathcal{B})_{cp})$$

where

$$\begin{array}{rcl} |\mathcal{A}| \oplus |\mathcal{B}| & & \text{is the coproduct in } \mathcal{C} \\ (\mathcal{A} \oplus \mathcal{B})_p & = & \mathcal{A}_p + \mathcal{B}_p \\ (\mathcal{A} \oplus \mathcal{B})_{cp} & = & \{\alpha + \beta \mid \alpha \in \mathcal{A}_{cp} \text{ and } \beta \in \mathcal{B}_{cp}\} \end{array}$$

Finally, the evident forgetful functor $U : \mathbf{G}\mathcal{C} \to \mathcal{C}$ is $*$-autonomous with left and right adjoints.