Z-modules and Full Completeness of Multiplicative Linear Logic

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Abstract

We prove that the full completeness theorem for $MLL + Mix$ holds by the simple interpretation via formulas as objects and proofs as Z-invariant morphisms in the *-autonomous category of topologized vector spaces. We do this by generalizing the recent work of R. F. Blute and P. J. Scott [8] where they used the semantical framework of dinatural transformation introduced by Girard-Scedrov-Scott [14]. By omitting the use of dinatural transformation, our semantics evidently allows the interpretation of the cut-rule, while the original Blute-Scott's does not. Moreover, our interpretation for proofs is preserved automatically under the cut elimination procedure. (In this sense, our semantics is considered as a denotational semantics.) In our semantics proofs themselves are characterized by the concrete algebraic notion "Z-invariance", and our denotational semantics provides the full completeness. Our semantics is naturally extended to the full completeness semantics for $CyLL + Mix$ thanks to an elegant method of Blute-Scott [9] (which is a sequel to [8]).

1 Introduction

R. Seely [22] showed that *-autonomous categories, which stemmed from M. Barr's discovery [3] of the category of topologized vector spaces $\mathcal{RTVEC}$, provided models for Linear Logic. By considering representations of Z (the additive group of integers) on the objects of $\mathcal{RTVEC}$, R.Blute and P.J.Scott [8] obtained the completeness theorem for Multiplicative Linear Logic with the Mix rule ($MLL + Mix$) using the semantical framework of dinatural transformation introduced by Girard-Scedrov-Scott [14]. Blute-Scott's completeness theorem is of a strong form called *full completeness theorem* (c.f. [2]); the mapping from the proofs to their interpretations is one to one and is linearly independent.

Blute-Scott [8] interpret formulas as multivariant functors (contravariant and covariant), and proofs as some kind of natural transformations between these functors, called *dinatural transformations*. The reason why multivariant functors (hence dinatural transformations) are required is that a subformula may occur simultaneously covariantly and contravariantly in a formula (e.g. $A \multimap A$).

But a weak point of their interpretation is, in the author's opinion, that dinatural transformations cannot allow composition in general; i.e, the composition of

*Research supported by Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists. This work was carried out while the author was at Department of Philosophy, Keio University (Tokyo).

dinatural transformations is not necessarily dinatural while that of natural transformations is. So a sequent derived by means of a cut rule cannot be interpreted in their interpretation. (Note that in a categorical model the cut rule corresponds to the composition of morphisms). Hence Blute-Scott [8] restrict their interpretation only to the cut free proofs as dinatural transformations (this is also the case of Girard-Scedrov-Scott [14]) and characterize a cut free proof as a Z-invariant dinatural transformation for the additive group Z of integers. It is very regrettable that in spite of the mathematically concrete structures of $\mathcal{RTVEC}$, their interpretations cannot accommodate a proof with cut-rules.

The starting point of this work is the impression that the categorical restriction "dinaturality" is very strong and that the Z-module structure of $\mathcal{RTVEC}$ do not seem to play an essential role in their full completeness proof. As a matter of fact, G. Plotkin pointed out that every dinatural transformation is Z-invariant (cf.Remark 3.3 of [9]).

Then a question arises whether the assumption of dinaturality can be omitted so that the notion of Z-invariant morphisms provides a semantics which allows the interpretation of cut-rules; moreover, whether such a semantics can be set up to be preserved under the cut-elimination reductions, hence can be considered as a denotational semantics. The purpose of this paper is to answer these questions affirmatively.

If we rely on a simple interpretation via proofs as morphisms, there exists no difficulty to obtain the soundness theorem. The difficulty with such a simple interpretation lies in the proof of the full completeness theorem which characterizes in concrete algebraic terms what kind of morphism $A^\bullet \to B^\bullet$ is the denotation of a proof for a sequent $A \vdash B$. Here, $A^\bullet$ and $B^\bullet$ stand for the interpretations of A and B into the objects of $\mathcal{RTVEC}$. In this paper, it is proved that any Z-invariant morphism is the denotation of some proof. This will be obtained by observing the fact that the category $\mathcal{RTVEC}$ has a natural algebraic structure although it is equipped with an abstract linear topology (cf. [7, 5]). Moreover our interpretation naturally ensures that the denotation of an arbitrary proof is preserved under cut-elimination, in fact, it is equal to that of a cut free proof. Hence our interpretation in this paper is considered as a denotational complete semantics for $MLL + Mix$ for "denotational" in the sense of Girard [12] p.25.

The contents of this paper are as follows: In Section 2, we shall demonstrate Soundness Theorem (Proposition 3), which is automatically obtained by our interpretation. The heart of this paper is Section 3 where the main results are;

Unique interpretation theorem for a binary simple sequent (Theorem 1)
$Hom_Z(A^\bullet, B^\bullet)$ is spanned by the interpretation of a unique (modulo reductions for normalization) proof of a binary simple sequent $A \vdash B$.

Completeness theorem for a binary simple sequent (Corollary 4)
If a binary simple sequent $A \vdash B$ is not provable, then $Hom_Z(A^\bullet, B^\bullet) = \mathbf{0}$

In the above Hom denotes the homset in the category $\mathcal{RTVEC}$ and $Hom_Z(A^\bullet, B^\bullet)$ denotes the subset consisting of the Z-invariant morphisms of $Hom(A^\bullet, B^\bullet)$: Note that $Hom(A^\bullet, B^\bullet)$ becomes a vector space, and so does $Hom_Z(A^\bullet, B^\bullet)$.

From these theorems the full completeness theorem for an arbitrary sequent is derived in the same way as in Blute-Scott [8];

Full Completeness Theorem (Proposition 7)
The vector space which is the denotation of the proofs of $M \vdash N$ has a basis consisting of the interpretations of the distinct (modulo reductions for normalization) proofs of $M \vdash N$.

In Section 4 we shall demonstrate that our semantics can be extended to a full completeness semantics for Cyclic Linear Logic with the Mix rule ($CyLL + Mix$) without any difficulty, thanks to an elegant method of Blute-Scott [9] (which is the sequel to their [8]).

In Section 2 and 3 we shall use the notations of Blute-Scott [8]. The reader is advised to refer to [8] for their detailed definitions. $\mathcal{TVEC}$ denotes the category whose objects are the vector spaces over discrete field $\mathbf{k}$ equipped with linear topologies (cf. [3, 21]) and whose morphisms are the linear continuous maps. $\mathcal{TVEC}$ becomes a symmetric monoidal closed category (autonomous category) with $(-) \otimes V$ defined as a left adjoint of $V \multimap (-)$ ($V \multimap W$ denotes the space of the linear continuous maps topologized by pointwise convergence). Note that the underlying vector space of $V \otimes W$ is the usual algebraic tensor product (cf. [7, 5]). Due to the linear topology, a canonical notion of duality of $\mathcal{TVEC}$, defined by $V^{\perp} := V \multimap \mathbf{k}$, has the nice property that the natural map $\mu : V \to V^{\perp\perp}$ is a bijection (cf. [3]). $\mathcal{RTVEC}$ denotes the subcategory consisting of the reflexive objects (i.e., μ is an isomorphism). $\mathcal{RTVEC}$ becomes a *-autonomous category (cf. [4]).

2 Soundness Theorem

Our goal in this section is the rather obvious soundness theorem. For this purpose, the category $\mathcal{RTMOD}(G)$ is introduced (Definition 4). Since all methods in this section follow just the same way as are described in Seely [22] in terms of the abstract category which axiomatizes linear logic, and in Girard [13] in terms of the mathematically concrete category of Banach spaces, the details of our interpretation of a proof as a morphism are moved to Appendix B. The reader familiar with Blute-Scott [8] may skip this section, and go to Section 3.

The language of Multiplicative Linear Logic with the Mix rule ($MLL + Mix$) consists of formulas and sequents. Formulas are generated from atoms (written by $X, Y, \ldots$) and constants $\mathbf{1}$ and $\perp$ by means of the binary connectives $\otimes$ and $\multimap$. The unary connective $\perp$ is defined by $A^{\perp} := A \multimap \perp$. A sequent is an expression $\Gamma \vdash \Delta$, where Γ and Δ are finite sequences of formulas. The provable sequents are generated by the inference rules of $MLL + Mix$ from the initial sequents (cf. Definition 13 of Appendix A for the inference rules of $MLL + Mix$).

Intuitionistic Multiplicative Linear Logic with the Mix rule ($IMLL + Mix$) is obtained from ($MLL + Mix$) by restricting the sequents to be of the form $\Gamma \vdash \Delta$ where Δ contains at most one formula.

Definition 1 (interpretation $A^{\bullet}$ of (I)MLL formula A into $(\mathcal{R})\mathcal{TVEC}$) *If each atom X is interpreted by an object $X^{\bullet}$ of $(\mathcal{R})\mathcal{TVEC}$ and the constants $\mathbf{1}$ and $\perp$ are interpreted by the base field $\mathbf{k}$ (i.e., $\mathbf{1}^{\bullet} := \perp^{\bullet} := \mathbf{k}$), then every formula A is interpreted by the object $A^{\bullet}$ of $(\mathcal{R})\mathcal{TVEC}$ according to the interpretation of connectives $\otimes$ and $\multimap$ by their corresponding functors of $(\mathcal{R})\mathcal{TVEC}$.*

Since the interpretation $A^{\bullet}$ of a formula A is uniquely determined by the interpretation of atoms occurring in A, we often write $A(X_1^{\bullet}, \ldots, X_n^{\bullet})$ for $A(X_1, \ldots, X_n)^{\bullet}$ where $X_1, \cdots, X_n$ are the list of the atoms occurring in A.

Definition 2 (interpretation $(\Gamma \vdash \Delta)^{\bullet}$ of sequent $\Gamma \vdash \Delta$) *For a sequence Γ of formulas $A_1, \ldots, A_n$, let $\Gamma^{\perp}$ denote the sequence $A_n^{\perp}, \ldots, A_1^{\perp}$ and let $\otimes\Gamma$ denote the formula $A_1 \otimes \cdots \otimes A_n$. A sequent $\Gamma \vdash \Delta$ is interpreted by the following homset in $(\mathcal{R})\mathcal{TVEC}$.*

$$Hom(((\otimes\Gamma) \otimes (\otimes\tilde{\Delta}^{\perp}))^{\bullet}, A^{\bullet}) \qquad \cdots (\star),$$

where $\Delta = A, \tilde{\Delta}$*, and if* Γ *(respectively,* Δ*) is empty we take* $\Gamma := \mathbf{1}$ *(respectively,* $A := \bot$*). The homset* $(\star)$ *is denoted by* $(\Gamma \vdash \Delta)^\bullet$*.*

Let $X_1, \ldots, X_n$ be the list of the atoms occurring in a sequent $\Gamma \vdash \Delta$. Then the $(\star)$ is often denoted by $Hom(B(X_1^\bullet, \ldots, X_n^\bullet), A(X_1^\bullet, \ldots, X_n^\bullet))$ where $B := (\otimes\Gamma) \otimes (\otimes\tilde{\Delta}^\bot)$.

The above interpretation of sequents naturally leads us to regard each proof rule as corresponding to the appropriate natural transformation between hom-functors. According to the correspondence, a proof π of a sequent $\Gamma \vdash \Delta$ is automatically interpreted by the morphism $\pi^\bullet$, where $\pi^\bullet \in (\Gamma \vdash \Delta)^\bullet$. (See Definition 15 of Appendix B for the detailed definition.)

Then it is evident that the interpretation is preserved under the cut elimination procedure:

Proposition 1 *If a proof* π *of a sequent* $\Gamma \vdash \Delta$ *normalizes into* π'*, then*

$$\pi^\bullet = (\pi')^\bullet.$$

Proof.
We can prove this easily by induction on the normalization steps from π to π'.
□

To obtain the full completeness theorem in the next section, we need a more concrete and stronger structure than $\mathcal{RTVEC}$. The appropriate structure is the representation of G on the objects of $\mathcal{RTVEC}$. This is provided by the category $\mathcal{RTMOD}(G)$ of G-modules in Blute-Scott [8]. In Definition 4 the category $\mathcal{TMOD}(G)$ and the subcategory $\mathcal{RTMOD}(G)$ are simultaneously defined for a fixed group G. We begin by reminding the reader of the definition of a representation of G:

Definition 3 (representation ρ of a group G) *For a group G and a vector space V, a representation ρ of G on V is a group homomorphism ρ*

$$\rho : G \to Aut(V),$$

where $Aut(V)$ denotes the group of automorphisms of V. To emphasize dependence on V, ρ is often denoted by (ρ, V). A representation (ρ, V) of G is identified with a G-module V. Hence we often write $g \cdot v$ for $\rho(g)(v)$.

Definition 4 (the category $(\mathcal{R})\mathcal{TMOD}(G)$ (cf. Blute-Scott [8]))
The category $(\mathcal{R})\mathcal{TMOD}(G)$ consists of the following:
*(***objects***): Representations (ρ, V) of G with V an object of $(\mathcal{R})\mathcal{TVEC}$ such that $\rho(g)$ is continuous for all $g \in G$.*
*(***morphisms***): The morphisms from (ρ, V) to (ρ', V') are all morphisms ϕ of $\mathcal{RTVEC}$ from V to V' such that*

$$\phi \circ \rho(g) = \rho'(g) \circ \phi \quad \textit{for all } g \in G.$$

For morphisms ϕ_1 and ϕ_2 of $(\mathcal{R})\mathcal{TMOD}(G)$, $\phi_1 \otimes \phi_2$ is defined by $(\phi_1 \otimes \phi_2)(u_1 \otimes u_2) := \phi_1(u_1) \otimes \phi_2(u_2)$. Then $\phi_1 \otimes \phi_2$ is a morphism of $(\mathcal{R})\mathcal{TMOD}(G)$. Thus $\otimes$ is a bifunctor on $(\mathcal{R})\mathcal{TMOD}(G)$.

Using the functors $\otimes$ and $\multimap$ of $\mathcal{TVEC}$, functors $\otimes$ and $\multimap$ in $\mathcal{TMOD}(G)$ can be defined as follows for given two objects (ρ, V) and (ρ', V');
*(***tensor product***):*

$$(\rho, V) \otimes (\rho', V') := (\rho \otimes \rho', V \otimes V'),$$

where $V \otimes V'$ is the tensor product of $\mathcal{TVEC}$ and $\sigma := \rho \otimes \rho'$ is defined by

$$(\sigma)(g)(v \otimes w) := \rho(g)(v) \otimes \rho'(g)(w).$$

(**exponential**):

$$(\rho, V) \multimap (\rho', V') := (\rho \multimap \rho', V \multimap V'),$$

where $V \multimap V'$ is the exponential of $\mathcal{TVEC}$ and $\sigma := \rho \multimap \rho'$ is defined by

$$f^{(\sigma)(g)}(-) := \rho'(g)(f(\ \rho(g^{-1})(-)).$$

Then the following becomes the unit for tensor product:
(**the unit for tensor product**):

$$(\mathbf{1}, \mathbf{k}),$$

where $\mathbf{1}$ is defined by

$$\mathbf{1}(g) \cdot k := k \quad \textit{for every } k \in \mathbf{k} \textit{ and every } g \in G.$$

Remark 1 *Let $\rho^{\perp}$ denote the representation $\rho \multimap \mathbf{1}$. Then the following is easily checked: For all $v' \in V^{\perp}$, $v \in V$ and $g \in G$,*

$$< \rho(g)(v'), \quad \rho^{\perp}(g)(v) >=< v', \ v >,$$

where $< \quad , \quad >$ denotes the natural bilinear form from V and $V^{\perp} := V \multimap \mathbf{k}$.

In the following we often identify a representation (ρ, V) with a G-module V. The following Remark 2 assures that $(\mathcal{R})\mathcal{TMOD}(G)$ becomes a (∗-)autonomous category with respect to the above functors and with the tensor unit as the dualizing object.

Remark 2 *Every natural isomorphism for establishing the *-autonomous structure of $\mathcal{TVEC}$ is an isomorphism of G-modules; namely, the following isomorphisms $a_{U,V,W}$, $s_{U,V}$, r_V, l_V and $\phi_{U,V,W}$ preserve the G-module structure for arbitrary G-modules whose underlying spaces are arbitrary objects U, V and W of $\mathcal{TVEC}$:*

$$a_{U,V,W} : (U \otimes V) \otimes W \tilde{\to} U \otimes (V \otimes W), \qquad s_{U,V} : U \otimes V \tilde{\to} V \otimes U,$$

$$r_V : V \otimes \mathbf{k} \tilde{\to} V, \qquad l_V : \mathbf{k} \otimes V \tilde{\to} V,$$

$$\phi_{U,V,W} : \mathcal{TVEC}(U \otimes V, W) \xrightarrow{\sim} \mathcal{TVEC}(V, U \multimap W).$$

As a consequence of the above, the following $\psi_{U,V}$ preserves the G-module structure for arbitrary G-modules whose underlying spaces are arbitrary objects U and V of $\mathcal{RTVEC}$:

$$\psi_{U,V} : \mathcal{RTVEC}(U, V) \xrightarrow{\sim} \mathcal{RTVEC}(V^{\perp}, U^{\perp}).$$

Proposition 2 (Blute-Scott [8])
$(\mathcal{R})\mathcal{TMOD}(G)$ is a (∗-)autonomous category with the tensor unit $(\mathbf{1}, \mathbf{k})$ as the dualizing object.

Proof.
From Remark 2 it is obvious that the natural isomorphisms for establishing the (*-)autonomous structure of $(\mathcal{R})\mathcal{TVEC}$ come to be those of $(\mathcal{R})\mathcal{TMOD}(G)$; especially if we restrict the above ϕ into $\mathcal{TVEC}(U \otimes V, W)^G$ (see below) then we obtain the following ϕ' which establishes the autonomous structure of $\mathcal{TMOD}(G)$:

$$\phi' : \mathcal{TMOD}(G)(U \otimes V, W) \quad \xrightarrow{\sim} \quad \mathcal{TMOD}(G)(V, U \multimap W),$$

where $\mathcal{TVEC}(U,V)^G := (U \multimap V)^G$ with W^G denoting the subspace $\{w \in W \mid g{\cdot}w = w\}$ of a G-module W. Note that $\mathcal{TMOD}(G)(U,V) = \mathcal{TVEC}(U,V)^G$ holds.
□

The $(I)MLL$ formulas are interpreted by the objects of $(\mathcal{R})\mathcal{TMOD}(G)$ as follows.

Definition 5 (the interpretation of a $(I)MLL$ formula A into $(\mathcal{R})\mathcal{TMOD}(G)$) *If each atom X is interpreted by an object $(\rho, X^\bullet)$ of $(\mathcal{R})\mathcal{TMOD}(G)$ and the constants $\mathbf{1}$ and $\perp$ are interpreted by the object $(\mathbf{1}, \mathbf{k})$, then every formula is interpreted by the unique object of $(\mathcal{R})\mathcal{TMOD}(G)$ according to the interpretation of connectives $\otimes$ and $\multimap$ by their corresponding functors of $(\mathcal{R})\mathcal{TMOD}(G)$, defined in Definition 4.*

Let $X_1, \cdots, X_n$ be the list of the atoms occurring in A. Since the interpretation of a formula $A(X_1, \ldots, X_n)$ into $(\mathcal{R})\mathcal{TMOD}(G)$ is uniquely determined by the interpretation $(\rho_i, X_i^\bullet)$ of the atoms X_i $(i = 1, \ldots, n)$, we write $A(\rho_1, \ldots, \rho_n)$ for the interpretation of a formula $A(X_1, \ldots, X_n)$; i.e., a formula $A(X_1, \ldots, X_n)$ is interpreted by the object $(A(\rho_1, \ldots, \rho_n), A(X_1^\bullet, \ldots, X_n^\bullet))$ of $(\mathcal{R})\mathcal{TMOD}(G)$. $A(\rho_1, \ldots, \rho_n)$ is called the *induced representation* by $\rho_1, \ldots, \rho_n$.

The interpretation of Definition 5 provides a very important property holding for certain morphisms of $(\mathcal{R})\mathcal{TVEC}$, called *$G$-invariance* (Definition 6), which characterizes a necessary condition for a morphism to be the interpretation of a proof (Proposition 3). On the other hand in Section 3, the property turns out to be a sufficient condition for a morphism to be the interpretation of a proof (cf. Theorem 1).

Definition 6 (G-invariant morphism:=G-map) *For a sequent $\Gamma \vdash \Delta$, a morphism $\phi \in (\Gamma \vdash \Delta)^\bullet := Hom(B(X_1^\bullet, \ldots, X_n^\bullet), A(X_1^\bullet, \ldots, X_n^\bullet))$ is said to be G-invariant if the following holds: For arbitrary representations ρ_i on $X_i^\bullet$ $(i = 1, \ldots, n)$ and $g \in G$, the following diagram commutes.*

$$\begin{tikzcd}
B(X_1^\bullet, \ldots, X_n^\bullet) \arrow[r, "\phi"] \arrow[d, "B(\rho_1, \ldots, \rho_n)(g)"'] & A(X_1^\bullet, \ldots, X_n^\bullet) \arrow[d, "A(\rho_1, \ldots, \rho_n)(g)"] \\
B(X_1^\bullet, \ldots, X_n^\bullet) \arrow[r, "\phi"'] & A(X_1^\bullet, \ldots, X_n^\bullet)
\end{tikzcd}$$

A G-invariant map is called a G-map.

Proposition 3 (soundness theorem of (I)MLL)
For any (I)MLL proof π of $\Gamma \vdash \Delta$, $\pi^\bullet$ is a G-map.

Proof:
From the following two facts we can check easily by induction on the construction of π;

- The interpretations of the initial sequents are G-maps (cf. Definition 15 in Appendix B).
- The natural isomorphisms corresponding to the inference rules are G-maps (cf. the above Remark 2).

□

3 Full Completeness Theorem

The main result in this section is Theorem 1 (the unique interpretation theorem for the binary simple sequents). Once Theorem 1 is proved, the completeness theorem for the binary simple sequents (Corollary 4) is derived as a corollary provided that the interpretation is *adequate* (Definition 9). The full completeness theorem for the arbitrary sequents (Proposition 7) is obtained from that for the binary sequents along the same line as Blute-Scott [8].

In the rest of this section, all representations considered are those of Z (the additive group of integers). In Definition 7 and Definition 8, we sum up the representations which we are going to use frequently. Note that these are all continuous (cf. [8]). In the following, $< v >$ denotes the vector space spanned by v over $\mathbf{k}$.

Definition 7 (representations p_{a} and p^{a}) *For a point* a *in a vector space* V *and a prime number* p*, the representations* p_{a} *of* Z *on* V *is defined as follows:*

$$(p_{\mathsf{a}}(n))(v) := \begin{cases} v & \textit{for } v \textit{ in} < a > \\ p^n \cdot v & \textit{for } v \textit{ in a complementary subspace of} < a >. \end{cases}$$

Note that $\{v \mid (p_{\mathsf{a}}(n))(v) = v \ \textit{ for all } n \in Z\}$ *is exactly the subspace* $< \mathsf{a} >$*. The representation* p^{a} *on* Z *is defined by* $p^{\mathsf{a}}(n) := (p_{\mathsf{a}}(n))^{-1} \circ (p_{\mathsf{0}}(n)) = (p_{\mathsf{0}}(n)) \circ (p_{\mathsf{a}}(n))^{-1}$ *where* $\mathsf{0}$ *denotes the origin of* V*.*

Definition 8 (representation $\sigma_{a,b}$ permuting a and b) *For linearly independent points* a_1 *and* $a_{(-1)}$ *from a vector space* V*, the representation* $\sigma_{a_1,a_{(-1)}}$ *of* Z *on* V *is defined as follows;*

$$(\sigma_{a_1,a_{-1}}(n))(v) := \begin{cases} a_{(-1)^n} & \textit{if } v = a_1 \\ a_{(-1)^{n+1}} & \textit{if } v = a_{(-1)} \\ v & \textit{for } v \textit{ in a complementary subspace of} < a_1, a_{(-1)} >. \end{cases}$$

Since a representation ρ of $\mathbf{Z}$ on V is uniquely determined by $\rho(1) \in Aut(V)$, in the following we often write ρ for $\rho(1)$ if no confusion is likely to occur.

In this section, our main task is as follows: For given two formulas A and B, we shall determine, by a purely algebraic method, the subset $Hom_Z(A^\bullet, B^\bullet)$ consisting of the Z-invariant morphisms of $Hom(A^\bullet, B^\bullet)$. So we recall the definition of a Z-map in the case where a given sequent is of the form $A \vdash B$ (Definition 6 with both Γ and Δ containing only one formula A and B, respectively). In what follows, $\alpha, \beta, \ldots$ denote interpretations $X^\bullet, Y^\bullet, \ldots$ of atoms $X, Y, \ldots$ into $\mathcal{RTVEC}$. For interpretations $A(\alpha_1, \ldots, \alpha_n)$ and $B(\alpha_1, \ldots, \alpha_n)$ of the two formulas into $\mathcal{RTVEC}$, a morphism ϕ of $\mathcal{RTVEC}$ from $A(\alpha_1, \ldots, \alpha_n)$ to $B(\alpha_1, \ldots, \alpha_n)$ is called a Z-map when for any representations ρ_i on α_i ($i = 1, \ldots, n$) and $1 \in Z$, the following diagram commutes.

$$\begin{tikzcd} A(\alpha_1, \ldots, \alpha_n) \arrow[r, "\phi"] \arrow[d, "A(\rho_1, \ldots, \rho_n)(1)"'] & B(\alpha_1, \ldots, \alpha_n) \arrow[d, "B(\rho_1, \ldots, \rho_n)(1)"] \\ A(\alpha_1, \ldots, \alpha_n) \arrow[r, "\phi"'] & B(\alpha_1, \ldots, \alpha_n) \end{tikzcd}$$

We denote the set of Z-maps from $A(\alpha_1, \ldots, \alpha_n)$ to $B(\alpha_1, \ldots, \alpha_n)$ by

$$Hom_Z(A(\alpha_1, \ldots, \alpha_n), B(\alpha_1, \ldots, \alpha_n)).$$

This becomes the vector space over $\mathbf{k}$ (under pointwise operations), which is the subspace of $Hom(A(\alpha_1, \ldots, \alpha_n), B(\alpha_1, \ldots, \alpha_n))$ with Hom denoting the homset in $\mathcal{RTVEC}$.

Proposition 4 *For any interpretation α of an atom,*

$$Hom_Z(\alpha, \alpha) =< \mathtt{id}_\alpha > .$$

Proof.
Take an arbitrary $\phi \in Hom_Z(\alpha, \alpha)$. For $a \in \alpha$,

$$\phi(a) = r_a \cdot a + v,$$

for some point v in a complementary subspace of $< a >$ and some $r_a \in \mathbf{k}$, because $\alpha \cong < a > \oplus \alpha / < a >$.

- Firstly, consider the representation $\rho := p^a$.

$$\begin{aligned} \phi^\rho(a) &= \rho(\phi(\rho^{-1}(a))) \\ &= \rho(\phi(p^{-n} \cdot a)) \\ &= p^{-n} \cdot \rho(\phi(a)) \\ &= p^{-n} \cdot \rho(r_a \cdot a + v) \\ &= p^{-n} \cdot (p^n \cdot r_a \cdot a + v) \\ &= r_a \cdot a + p^{-n} \cdot v \end{aligned}$$

  $\phi = \phi^\rho$ implies $v = 0$. Hence $\phi(a) = r_a \cdot a$.

- Secondly, we show $r_a = r_b$ for arbitrary a and b in α. Consider the representation $\sigma := \sigma_{a,b}$ for $a, b \in \alpha$. Then

$$\phi^\sigma(a) = r_b \cdot a$$

  $\phi = \phi^\sigma$ implies $r_a = r_b$. Hence $\phi = r \cdot \mathtt{id}_\alpha$ with some $r \in \mathbf{k}$.

□

Corollary 1 *For any interpretation α of an atom,*

$$Hom_Z(\alpha^\perp, \alpha^\perp) =< \mathtt{id}_\alpha^\perp > .$$

Proof.
From Remark 2 of Section 2.
□

In the following, the interpretations to be considered are those of binary sequents (cf. [2, 8]), namely those sequents where each atom occurs exactly twice, once positively and once negatively; note that $\alpha_1, \ldots, \alpha_n$ and β in the following may be an identical object of $\mathcal{RTVEC}$, but the representations on $\alpha_1, \ldots, \alpha_n$ and on β run independently over $\mathcal{RTMOD}(Z)$.

Corollary 2 *For any interpretation α and β of two atoms,*

$$Hom_Z(\alpha \otimes \alpha^\perp \otimes \beta, \beta) = << \ , \ > \cdot \mathtt{Id}_\beta > \tag{1}$$

Proof.
For ϕ from the left hand-side of (1) and $a \otimes a' \in \alpha \otimes \alpha^{\perp}$, the linear map $\phi_{a,a'} \in (\beta \multimap \beta)$ is defined by $\phi_{a,a'}(b) := \phi(a \otimes a' \otimes b)$. Because ϕ is a Z-map, so is $\phi_{a,a'}$. From Proposition 4, $\phi_{a,a'} = r(a,a') \cdot \mathtt{id}_{\beta}$ with $r(a,a') \in \mathbf{k}$. Since ϕ is a Z-map, so is $r(\ ,\ )$. Then from Proposition 4 and Remark 2, $r(\ ,\ )$ is a scalar multiple of $<\ ,\ >$.
□

In the following $\alpha^{\perp} \otimes \beta$ is often identified with the subspace of $\alpha \multimap \beta$ through the canonical monomorphism $\mathtt{mix} : \alpha^{\perp} \otimes \beta \hookrightarrow \alpha \multimap \beta$ with $\mathtt{mix}$ defined by $\mathtt{mix}(a' \otimes b) :=< a', - > \cdot\ b$. By virtue of the identification, the right hand-side of (1) in Corollary 2 is identified with $< \mathtt{ev}'_{\alpha,\beta} >$, where $\mathtt{ev}'_{\alpha,\beta}$ is $\mathtt{ev}_{\alpha,\beta}$ with its domain restricted to the subspace.

Obviously $\mathtt{mix}$ is a Z-invariant monomorphism. Moreover the following holds:

Corollary 3 (uniqueness of Mix) *For any interpretation α and β of two atoms,*

$$Hom_Z(\alpha^{\perp} \otimes \beta, \alpha \multimap \beta) =< \mathtt{mix} >,$$

Proof.
From Corollary 2 and Remark 2.
□

Proposition 5 (base case for unique interpretation theorem) *For any interpretation α and β of two atoms,*

$$Hom_Z(\alpha \otimes (\alpha \multimap \beta), \beta) =< \mathtt{ev}_{\alpha,\beta} > . \tag{2}$$

Proof. Take arbitrary $a \in \alpha$ and $f \in (\alpha \multimap \beta)$ such that $f(a) \neq 0$. Let $\{c_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ denote bases of complementary subspaces of a in α and of $f(a)$ in β, respectively. Since f is linear, it is determined by:

$$f(c_i) \quad = \quad r_i \cdot f(a) + \sum_{j \in J} r_{ji} \cdot b_j \quad \text{where } i \in I.$$

Consider the induced representation ρ from the representations p_a and $p_{f(a)}$ on α and β, respectively. Then f^{ρ} is:

$$\begin{aligned} f^{\rho}(c_i) &= p^{-1} \cdot r_i \cdot f(a) + \sum_{j \in J} r_{ji} \cdot b_j \quad \text{where } i \in I, \\ f^{\rho}(a) &= f(a). \end{aligned}$$

Hence $(f - f^{\rho})$ is:

$$(f - f^{\rho})(c_j) \quad = \quad (1 - p^{-1}) \cdot r_i \cdot f(a), \tag{3}$$

$$(f - f^{\rho})(a) \quad = \quad 0. \tag{4}$$

Then (3) and (4) imply that

$$(f - f^{\rho}) \in \alpha^{\perp} \otimes \beta. \tag{5}$$

Now take an arbitrary ϕ from the left hand-side of (2). Note that (5) together with Corollary 2 implies that $\phi(a \otimes (f - f^{\rho})) = k \cdot \mathtt{ev}'(a \otimes (f - f^{\rho})) = k \cdot (f - f^{\rho})(a)$ (cf. the paragraph after Corollary 2). Then

$$\begin{aligned} \phi((a \otimes f) - \rho(a \otimes f)) &= \phi((a \otimes f) - (a \otimes f^{\rho})) && \text{since } \rho(a) = a \\ &= \phi(a \otimes (f - f^{\rho})) \\ &= k \cdot (f - f^{\rho})(a) && \text{from (5) and Corollary 2} \\ &= k \cdot (f(a) - f^{\rho}(a)) \\ &= 0 && \text{since } a \text{ and } f(a) \text{ are fixed by } \rho. \end{aligned}$$

Since ϕ is a Z-map, the above implies;

$$\phi(a \otimes f) \quad = \quad \rho(\phi(a \otimes f)).$$

Since the fix points in β by ρ are the scalar multiples of $f(a)$,

$$\phi(a \otimes f) \quad = \quad k \cdot f(a) \qquad\qquad \text{with some scalar } k.$$

Let $k(a, f)$ denote the scalar k determined by a and f in the above way. We have to show that $k(a, f) = k(a', f')$ for every $a, a' \in \alpha$ and for every $f, f' \in (\alpha \multimap \beta)$.

(a) Fix $f \in (\alpha \multimap \beta)$. Then for arbitrary $a, a' \in \alpha$ such that $f(a)$ and $f(a')$ are linearly independent, $k(a, f) = k(a', f)$ holds since $\phi(a \otimes f) + \phi(a' \otimes f) = \phi((a + a') \otimes f)$.

(b) Fix $a \in \alpha$. Then for arbitrary $f, f' \in \alpha$ such that $f(a)$ and $f'(a)$ are linearly independent, $k(a, f) = k(a, f')$ holds since $\phi(a \otimes f) + \phi(a \otimes f') = \phi(a \otimes (f + f'))$.

From (a) and (b), it follows that $k(a, f) = k(a', f')$ for every $a, a' \in \alpha$ and for every $f, f' \in (\alpha \multimap \beta)$.
$\square$

A sequent $\vdash \Gamma$ is simple if all of the formulas of Γ are literals or a tensor product of two literals (cf. [2, 8]). It is observed in [8] that all connected components (in the sense of the proof structure representation for proofs) of a provable binary simple sequent in $MLL + Mix$ must be of the form;

$$X_1 \otimes (X_1 \multimap X_2) \otimes \cdots \otimes (X_n \multimap X_{n+1}) \vdash X_{n+1}.$$

Hence the following is the unique interpretation theorem for a binary simple sequent, where the Z-map $\mathtt{ev}_{\alpha_1,\ldots,\alpha_{n+1}}$ is defined inductively as follows;

$$\mathtt{ev}_{\alpha_1,\ldots,\alpha_n,\alpha_{n+1}} := \mathtt{ev}_{\alpha_n,\alpha_{n+1}} \circ (\mathtt{ev}_{\alpha_1,\ldots,\alpha_n} \otimes \mathtt{id}_{\alpha_n \multimap \alpha_{n+1}}).$$

Theorem 1 (unique interpretation theorem for a binary simple sequent) *For any interpretation $\alpha_1, \ldots, \alpha_{n+1}$ of atoms $X_1, \ldots, X_{n+1}$,*

$$Hom_Z(\alpha_1 \otimes (\alpha_1 \multimap \alpha_2) \otimes \cdots \otimes (\alpha_n \multimap \alpha_{n+1}), \alpha_{n+1}) = < \mathtt{ev}_{\alpha_1,\ldots,\alpha_{n+1}} > . \qquad (6)$$

Proof. The theorem for $n = 1$ is Proposition 5. We shall prove by induction on n. We define $F(\alpha_1, \ldots, \alpha_{n+1})$ and the subspace $F_j(\alpha_1, \ldots, \alpha_{n+1})$ of F for $1 \leq j \leq n$ as follows;

$$\begin{aligned} F &:= \alpha_1 \otimes (\alpha_1 \multimap \alpha_2) \otimes \cdots \otimes (\alpha_{j-1} \multimap \alpha_j) \otimes (\alpha_j \multimap \alpha_{j+1}) \otimes (\alpha_{j+1} \multimap \alpha_{j+2}) \otimes \cdots \otimes (\alpha_n \multimap \alpha_{n+1}), \\ F_j &:= \alpha_1 \otimes (\alpha_1 \multimap \alpha_2) \otimes \cdots \otimes (\alpha_{j-1} \multimap \alpha_j) \otimes (\alpha_j^\perp \otimes \alpha_{j+1}) \otimes (\alpha_{j+1} \multimap \alpha_{j+2}) \otimes \cdots \otimes (\alpha_n \multimap \alpha_{n+1}). \end{aligned}$$

Note that $F(\alpha_1, \ldots, \alpha_j) \otimes \alpha_j^\perp \otimes F(\alpha_{j+1}, \ldots, \alpha_{n+1}) = F_j(\alpha_1, \ldots, \alpha_{n+1})$.
From the induction hypothesis for every j with $1 < j < n$, the following (7) and (8) hold;

$$\begin{aligned} Hom_Z(F(\alpha_1, \ldots, \alpha_j), \alpha_j) &= < \mathtt{ev}_{\alpha_1,\ldots,\alpha_j} >, & (7) \\ Hom_Z(F(\alpha_{j+1}, \ldots, \alpha_{n+1}), \alpha_{n+1}) &= < \mathtt{ev}_{\alpha_{j+1},\ldots,\alpha_{n+1}} > . & (8) \end{aligned}$$

(7) implies (9) by Remark 2;

$$Hom_Z(F(\alpha_1, \ldots, \alpha_j) \otimes \alpha_j^\perp, \mathbf{k}) \quad = \quad << \ , \ > \circ (\mathtt{ev}_{\alpha_1,\ldots,\alpha_j} \otimes \mathtt{id}_{\alpha_j^\perp}) > . \quad (9)$$

We begin by proving the equation (10) below with the help of (8) and (9) above;

$$Hom_Z(F_j(\alpha_1, \ldots, \alpha_{n+1}), \alpha_{n+1}) \quad = \quad < \mathtt{ev}^j_{\alpha_1,\ldots,\alpha_{n+1}} >, \qquad (10)$$

where $\mathtt{ev}^j_{\alpha_1,\ldots,\alpha_{n+1}}$ is $\mathtt{ev}_{\alpha_1,\ldots,\alpha_{n+1}}$ with its domain restricted to the subspace $F_j(\alpha_1,\ldots,\alpha_{n+1})$ of $F(\alpha_1,\ldots,\alpha_{n+1})$; i.e., $\mathtt{ev}^j_{\alpha_1,\ldots,\alpha_{n+1}}(a_1\otimes f_1\otimes\cdots\otimes f_{j-1}\otimes a'_j\otimes a_{j+1}\otimes f_{j+1}\cdots\otimes f_n) = f_n(\cdots f_{j+1}(< a'_j, f_{j-1}(\cdots f_1(a_1)\cdots) > \cdot a_{j+1})\cdots)$.
For an arbitrary ϕ^j from the left hand-side of (10) and an arbitrary $c\in F(\alpha_1,\ldots,\alpha_j)\otimes\alpha_j^\perp$, $\phi^j_c \in \mathit{Hom}(F(\alpha_{j+1},\ldots,\alpha_{n+1}),\alpha_{n+1})$ is defined by $\phi^j_c(d) := \phi^j(c\otimes d)$ with $d\in F(\alpha_{j+1},\ldots,\alpha_{n+1})$. Then ϕ^j_c is a Z-map. Hence from the induction hypothesis (8), $\phi^j_c = k(c)\cdot\mathtt{ev}_{\alpha_{j+1},\ldots,\alpha_{n+1}}$ with some scalar $k(c)$ determined by c. On the other hand, $k(-)\in \mathit{Hom}(F(\alpha_1,\ldots,\alpha_j)\otimes\alpha_j^\perp,\mathbf{k})$ is also a Z-map. Hence from the induction hypothesis (9), $k(-) = <\ ,\ > \circ(\mathtt{ev}_{\alpha_1,\ldots,\alpha_j}\otimes\mathtt{id}_{\alpha_j^\perp})(-)$. Therefore the following holds for some scalar r_j;

$$\begin{aligned}
&\phi_j(a_1\otimes f_1\otimes\cdots\otimes f_{j-1}\otimes a'_j\otimes a_{j+1}\otimes f_{j+1}\cdots\otimes f_n)\\
&= r_j\cdot < a'_j, f_{j-1}(\cdots f_1(a_1)\cdots) > \cdot f_n(\cdots f_{j+1}(a_{j+1})\cdots)\\
&= r_j\cdot f_n(\cdots f_{j+1}(< a'_j, f_{j-1}(\cdots f_1(a_1)\cdots) > \cdot a_{j+1})\cdots).
\end{aligned}$$

This completes the proof of the equation (10); $\phi^j = r_j\cdot\mathtt{ev}^j_{\alpha_1,\ldots,\alpha_{n+1}}$.

The equation (10) means that for an arbitrary ϕ from the left hand-side of (6), its restriction on F_j is a scalar multiple of $\mathtt{ev}^j_{\alpha_1,\ldots\alpha_{n+1}}$. Namely, for an arbitrary Z-map ϕ and for every j with $1\le j\le n$, the following diagram commutes (modulo scalar multiples);

$$\begin{tikzcd}
F_j(\alpha_1,\ldots,\alpha_{n+1}) \arrow[d, hook] \arrow[dr, "\mathtt{ev}^j_{\alpha_1,\ldots,\alpha_{n+1}}"] & \\
F(\alpha_1,\ldots,\alpha_{n+1}) \arrow[r, "\phi"] & \alpha_{n+1}
\end{tikzcd}$$

Now for a given $a\otimes f_1\otimes\cdots\otimes f_n\in F$, consider the representations p_{b_j} on α_j where $b_j := f_{j-1}(\cdots f_1(a)\cdots)$ with $j=1,\ldots,n+1$. Let ρ denote the induced representation by p_{b_j}. Remember that the fix points by ρ in α_j are the scalar multiples of b_j.
With the help of the algebraic equation (11) below

$$\begin{aligned}
&(x_1\otimes\cdots\otimes x_n) - (y_1\otimes\cdots\otimes y_n)\\
&= \sum_{j=1}^n \{x_1\otimes\cdots\otimes x_{j-1}\otimes(x_j-y_j)\otimes y_{j+1}\otimes\cdots\otimes y_n\},
\end{aligned}\tag{11}$$

the following holds;

$$\begin{aligned}
& (a\otimes f_1\otimes\cdots\otimes f_n) - \rho(a\otimes f_1\otimes\cdots\otimes f_n) && \\
= {} & a\otimes((f_1\otimes\cdots\otimes f_n) - (f_1^\rho\otimes\cdots\otimes f_n^\rho)) && \text{since } \rho(a)=a\\
= {} & a\otimes\{\sum_{j=1}^n f_1\otimes\cdots\otimes f_{j-1}\otimes(f_j-f_j^\rho)\otimes f_{j+1}^\rho\otimes\cdots\otimes f_n^\rho\} && \text{from (11)}\\
= {} & \sum_{j=1}^n a\otimes f_1\otimes\cdots\otimes f_{j-1}\otimes(f_j-f_j^\rho)\otimes f_{j+1}^\rho\otimes\cdots\otimes f_n^\rho. && (12)
\end{aligned}$$

By the same argument as in the proof of Proposition 5 (cf.equations (3) and (4)),

$$(f_j - f_j^\rho)\in\alpha_j^\perp\otimes\alpha_{j+1}.$$

Hence

$$a\otimes f_1\otimes\cdots\otimes f_{j-1}\otimes(f_j-f_j^\rho)\otimes f_{j+1}^\rho\otimes\cdots\otimes f_n^\rho\in F_j. \tag{13}$$

Take an arbitrary ϕ from the left hand-side of (6). Then (12) implies the following;

$$\begin{array}{rl}
& \phi((a \otimes f_1 \otimes \cdots \otimes f_n) - \rho(a \otimes f_1 \otimes \cdots \otimes f_n)) \\
= & \displaystyle\sum_{j=1}^{n} \phi(a \otimes f_1 \otimes \cdots \otimes f_{j-1} \otimes (f_j - f_j^\rho) \otimes f_{j+1}^\rho \otimes \cdots \otimes f_n^\rho) \\
= & \displaystyle\sum_{j=1}^{n} r_j \cdot \mathtt{ev}^j(a \otimes f_1 \otimes \cdots \otimes f_{j-1} \otimes (f_j - f_j^\rho) \otimes f_{j+1}^\rho \otimes \cdots \otimes f_n^\rho) \begin{array}{r}\text{from (13) and (10)} \\ \text{with some scalars } r_j\end{array} \\
= & \displaystyle\sum_{j=1}^{n} r_j \cdot (f_n^\rho \circ \cdots \circ f_{j+1}^\rho \circ (f_j - f_j^\rho) \circ f_{j-1} \circ \cdots \circ f_1)(a) \\
= & \displaystyle\sum_{j=1}^{n} r_j \cdot \{(f_n^\rho \circ \cdots \circ f_{j+1}^\rho \circ f_j \circ f_{j-1} \circ \cdots \circ f_1)(a) - (f_n^\rho \circ \cdots \circ f_{j+1}^\rho \circ f_j^\rho \circ f_{j-1} \circ \cdots \circ f_1)(a)\} \\
= & 0 \qquad\qquad\qquad \text{from the definition of } \rho.
\end{array}$$

I.e., $\phi(a \otimes f_1 \otimes \cdots \otimes f_n)$ is fixed by ρ.

Since the fix points in α_{n+1} by ρ are the scalar multiples of $f_n(\cdots f_1(a) \cdots)$, i.e., $\phi(a \otimes f_1 \otimes \cdots \otimes f_n) = r(a, f_1, \ldots, f_n) \cdot f_n(\cdots f_1(a) \cdots)$ with some scalar $r(a, f_1, \ldots, f_n) \in \mathbf{k}$ determined by $a \otimes f_1 \otimes \cdots \otimes f_n$. The uniqueness of $r(a, f_1, \ldots, f_n)$ is proved by the same manner as in the proof of Proposition 5. We have thus proved Theorem 1; ϕ is a scalar multiple of $\mathtt{ev}_{\alpha_1, \ldots, \alpha_{n+1}}$.
□

It is observed in Blute-Scott [8] that any unprovable binary simple sequent must be of the form;

$$\Gamma, (X_1 \multimap X_2) \otimes \cdots \otimes (X_n \multimap X_{n+1}) \vdash X_1^\perp \otimes X_{n+1}.$$

Then the following Corollary 4 of Theorem 1 is the completeness theorem for a binary simple sequent. Before going to the corollary, we prepare the following definition:

Definition 9 (adequate interpretation) *Let $\mathcal{RTVEC}_\infty$ denote the subcategory $\mathcal{RTVEC}$ consisting of the objects whose dimension over $\mathbf{k}$ is infinite. We call an interpretation $\{\alpha_i\}_{i=1}^n$ into $\mathcal{RTVEC}_\infty$ of atoms $\{X_i\}_{i=1}^n$ adequate if for all i and j, the following holds:*

$$(\alpha_i^\perp \otimes \alpha_j) \xhookrightarrow{\neq} (\alpha_i \multimap \alpha_j).$$

For example $\{\alpha_i\}_{i=1}^n := \{\alpha\}$ is adequate because $\mathtt{id}_\alpha \in \alpha \multimap \alpha$ but $\mathtt{id}_\alpha \notin \alpha^\perp \otimes \alpha$

Corollary 4 (completeness theorem for a binary simple sequent) *For any adequate interpretation $\alpha_1, \ldots, \alpha_{n+1}$ of atoms $X_1, \ldots, X_{n+1}$*

$$Hom_Z((\otimes\Gamma)^\bullet \otimes (\alpha_1 \multimap \alpha_2) \otimes \cdots \otimes (\alpha_n \multimap \alpha_{n+1}), \alpha_1^\perp \otimes \alpha_{n+1}) = 0. \qquad (14)$$

Proof. For arbitrary ϕ from the left hand-side of (14) and $\gamma \in (\otimes\Gamma)^\bullet$ the linear map $\phi_\gamma \in Hom((\alpha_1 \multimap \alpha_2) \otimes \cdots \otimes (\alpha_n \multimap \alpha_{n+1}), \alpha_1^\perp \otimes \alpha_{n+1})$ is defined by $\phi_\gamma(x) := \phi(\gamma \otimes x)$. Because ϕ is a Z-map, so is ϕ_γ. Hence it suffices in the following to prove the assertion when Γ is empty. For an arbitrary ϕ from the left hand-side of (14) with Γ empty, Theorem 1 and Remark 2 imply that $(\mathtt{mix}) \circ \phi$ must be a scalar multiple of the natural composition $\mathtt{comp}_{\alpha_1, \ldots, \alpha_n}$: For an arbitrary Z-map ϕ

the following diagram commutes (modulo scalar multiples);

$$\begin{tikzcd}
(\alpha_1 \multimap \alpha_2) \otimes \cdots \otimes (\alpha_n \multimap \alpha_{n+1}) \arrow[r, "\phi"] \arrow[dr, "\mathtt{comp}_{\alpha_1,\ldots,\alpha_n}"'] & \alpha_1^{\perp} \otimes \alpha_{n+1} \arrow[d, hook, "\mathtt{mix}"] \\
& \alpha_1 \multimap \alpha_{n+1}
\end{tikzcd}$$

From the assumption that the interpretation is adequate, the only possible case is the case where $\phi = 0$.
□

The following Proposition 6 is obtained along the same line as Blute-Scott [8] from Theorem 1 and Corollary 4 and by using the fact that the `mix` and the following morphisms (called weak distributivity) in $\mathcal{TVEC}$ are monic: (i)$(U \otimes (V \multimap W)) \to ((U \multimap V) \multimap W)$ (ii)$(U \otimes (V \multimap W)) \to ((U \multimap V) \multimap W)$.

Proposition 6 (full completeness theorem for a binary sequent) *For a binary sequent*

$$A(X_1, \ldots, X_n) \vdash B(X, \ldots, X_n) \tag{15}$$

and an adequate interpretation $\{\alpha_i\}_{i=1}^n$ *of* $\{X_i\}_{i=1,\ldots,n}$ *into* $\mathcal{RTVEC}_\infty$*,*

$$\mathtt{dim}_{\Bbbk}(Hom_Z(A(\alpha_1, \ldots, \alpha_n), B(\alpha, \ldots, \alpha_n))) = \begin{cases} \mathtt{0} & \textit{if (15) is not provable} \\ \mathtt{1} & \textit{if (15) is provable} \cdots (\star). \end{cases}$$

Note that $(\star)$ of Proposition 6 holds without the requirement that an interpretation considered is adequate. The above full completeness theorem contains the faithfulness of our interpretation because any binary provable sequent has a unique (modulo reductions for normalization) proof.

For an arbitrary sequent $M \vdash N$, let $M_1 \vdash N_1, \ldots, M_n \vdash N_n$ be the list of the binary sequents of which $M \vdash N$ is a substitution instance. Every proof of a sequent $M \vdash N$ results from that of $M_i \vdash N_i$ by equating atoms appropriately. Hence the associated binary space $\mathcal{ABS}(M, N) := \coprod_{i=1}^n Hom_Z(M_i \vdash N_i)$ of Blute-Scott's (cf. [8]) is a natural denotation for the proofs of $M \vdash N$. Then Proposition 6 implies the following:

Proposition 7 (full completeness theorem for an arbitrary sequent)
$\mathcal{ABS}(M, N)$ has a basis consisting of interpretations of all distinct (modulo reductions for normalization) proofs of $M \vdash N$.

Proposition 7 can be restated as the following fully faithful representation theorem (Proposition 7') analogously to Remark 3.16 of Blute-Scott [9]: Let $\mathcal{F}$ denote the free *-autonomous category with Mix on a discrete graph of atoms and $LIN(\mathcal{F})$ denote the category whose objects are the same as in $\mathcal{F}$, but whose homsets are the linear combinations of morphisms from the corresponding homsets in $\mathcal{F}$. Let φ-$\mathcal{RTVEC}$ denote the category whose objects are same as in $\mathcal{RTVEC}$ but whose homset from M to N is the image of the canonical linear map $\varphi : \mathcal{ABS}(M, N) \to \mathcal{RTVEC}(M, N)$. Note that in contrast to Blute-Scott [8], φ-$\mathcal{RTVEC}$ automatically becomes a *-autonomous category independently of our full completeness theorem (cf. Corollary 10.10 of [8]). Now our fully faithful representation theorem is the following:

Proposition 7' (fully faithful representation theorem)
Every free functor $(\ )^\bullet : LIN(\mathcal{F}) \to \varphi$-$\mathcal{RTVEC}$ is fully faithful (i.e., an equivalence) provided that $(\ )^\bullet$ is adequate.

4 Full Completeness Theorem for Cyclic Linear Logic with Mix

In this section we shall explain an extension of our work to Cyclic Linear Logic with the Mix rule ($CyLL + Mix$) along the line of Blute-Scott [9], where they proved the full completeness theorem for $CyLL + Mix$ by generalizing the representation of a group to that of a Hopf algebra with an involutive antipode (in particular, by generalizing the representation of the additive group Z of integers to that of the shuffle Hopf algebra $\mathcal{A}$). (See Appendix A for the syntax of $CyLL + Mix$.) In contrast to their full completeness proof of [8] for $MLL + Mix$ where they impose the dinaturality condition, their proof of the completeness theorem for a noncyclic sequent for $CyLL + Mix$ (Section 9.1 in [9]) does not impose the dinaturality condition but imposes only the $\mathcal{A}$-invariance. Hence, their proof (in [9]) can be adjusted to our frame work (introduced in former sections) directly. In this section we shall review this in Proposition 11 and Proposition 12. Combined Proposition 11 and Proposition 12 with our full completeness theorem for $MLL + Mix$ in Section 3, the full completeness theorem for $CyLL + Mix$ is automatically obtained.

Blute-Scott [9] is based on pioneering work of Blute [7], where he discovered that a categorical model of $CyLL+Mix$ was obtained by considering representations of a Hopf algebra with a bijective antipode on the objects of $\mathcal{RTVEC}$. Let us begin with reviewing Blute [7]. For the precise definition of a Hopf algebra, see [7]. All notations in this section follow [7]. A Hopf algebra H is denoted by $< H, m, \eta, \Delta, \epsilon, S >$, where (i) $< H, m, \eta >$ is an algebra with m a multiplication and η a unit, (ii) $< H, \Delta, \epsilon >$ is a coalgebra with Δ a comultiplication and ϵ a counit, (iii)m and η are maps of coalgebras, (equivalently, Δ and ϵ are maps of algebras), and (iv)S is an antipode; namely, $m \circ (S \otimes \mathtt{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathtt{id} \otimes S) \circ \Delta$. In this section, the following notations are used; for an algebra $A :=< A, m, \eta >$, $m(a \otimes b)$ is denoted by ab; for a coalgebra $B :=< B, \delta, \epsilon >$, $\Delta(b)$ is denoted by $\Sigma b_1 \otimes b_2$, which is known as the Sweedler's sigma notation (cf. [24]). Since $(\mathtt{id} \otimes \Delta) \circ \Delta(h) = (\Delta \otimes \mathtt{id}) \circ \Delta(h)$ holds by the coassociativity of Δ, the element is denoted by $\Sigma h_1 \otimes h_2 \otimes h_3$.

Definition 10 (A representation ρ of an algebra A) *For an algebra $A :=< A, m, \eta >$ and a vector space V, a representation ρ of A on V is an algebra homomorphism*

$$\rho : A \to End(V),$$

where $End(V)$ denotes the algebra of endomorphisms of V. To stress V, ρ is often denoted by (ρ, V). A representation (ρ, V) of H is identified with an H-module V. Hence we often write $h \cdot v$ for $\rho(h)(v)$.

A representation of an algebra A on V is identified with an A-module V. A representation of a Hopf algebra H is a representation of the algebra structure of H. For a Hopf algebra H, let H^{cop} denote the bialgebra $< H, m, \eta, \Delta^{op}, \epsilon >$, where $\Delta^{op}(h) := \Sigma h_2 \otimes h_1$. Since the algebra structure of H^{cop} is the same as that of H, a representation of H^{cop} is identified with that of H. If the antipode S of H is bijective, then H^{cop} becomes a Hopf algebra with the antipode S^{-1}. Now let us define Blute's category $(\mathcal{R})\mathcal{TMOD}(H)$ (cf. [7]) of representations of a Hopf algebra H on objects of $(\mathcal{R})\mathcal{TVEC}$, which becomes a non symmetric monoidal closed category provided that the antipode S of H is bijective. From now on, H denotes a Hopf algebra with bijective antipode. The crucial point to obtain the closed structure of the nonsymmetrical monoidal category $(\mathcal{R})\mathcal{TMOD}(H)$ is the following; for given algebra A and A-modules V and W, $V \multimap W$ has the natural $A \otimes A^{op}$-module structure given by $f^{a \otimes a'}(-) := af(a'-)$ where $f \in V \multimap W$, $a \otimes a' \in A \otimes A^{op}$ and $A^{op} :=< A, m^{op}, \eta >$ with $m^{op}(b \otimes c) := m(c \otimes b)$. Now there exist two algebra homomorphisms from H to $H \otimes H^{op}$; one is $(\mathtt{id} \otimes S) \circ \Delta$ and the other is

$(\mathtt{id} \otimes S^{-1}) \circ \Delta^{op}$; the former (respectively, the latter) is obtained by using the Hopf algebra structure of H (respectively, H^{cop}). By using the two homomorphisms, one can define two kinds of representation on $V \multimap W$ of H from given representations (ρ, V) and (τ, W) (cf. see exponential representations $\circ\!-$ and $\multimap$ in the following definition).

Definition 11 (the $(\mathcal{R})\mathcal{TMOD}(H)$ (cf. Blute [7] and Blute-Scott [9])) *For a Hopf algebra H with a bijective antipode S, the category $(\mathcal{R})\mathcal{TMOD}(H)$ consists of the following:*

(objects): *Representations (ρ, V) of Hopf algebra H with V an objects of $(\mathcal{R})\mathcal{TVEC}$ such that $\rho(h)$ is continuous for every $h \in H$.*

(morphisms): *The morphisms from (ρ, V) to (ρ', V') are all morphisms ϕ of $\mathcal{TVEC}$ from V to V' such that*

$$\rho'(h) \circ \phi = \phi \circ \rho(h) \qquad \text{for all } h \in H.$$

For morphisms ϕ_1 and ϕ_2 of $(\mathcal{R})\mathcal{TMOD}(H)$, $\phi_1 \otimes \phi_2$ is defined by $(\phi_1 \otimes \phi_2)(u_1 \otimes u_2) := \phi_1(u_1) \otimes \phi_2(u_2)$. Then $\phi_1 \otimes \phi_2$ is a morphism of $(\mathcal{R})\mathcal{TMOD}(H)$. Thus $\otimes$ is a bifunctor on $(\mathcal{R})\mathcal{TMOD}(H)$.

By virtue of Δ, tensor representations can be defined as follows (note that $\Delta(h)$ is written as $\Sigma h_1 \otimes h_2$ in the following):

(tensor product):

$$(\rho, V) \otimes (\tau, W) := (\rho \otimes \tau, V \otimes W),$$

where $\rho \otimes \tau$ is defined by

$$(\rho \otimes \tau)(h)(v \otimes w) := \sum_{(h)} \rho(h_1)(v) \otimes \tau(h_2)(w).$$

By using the coalgebra and the antipode for H and these for H^{cop}, respectively, exponential representations $\circ\!-$ and $\multimap$ can be defined as follows: Note that a representation of H^{cop} is identified with that of H because the algebra structure of H^{cop} is identical to that of H.

(exponential $\circ\!-$): *By using the fact that $(\mathtt{id} \otimes S) \circ \Delta$ is an algebra homomorphism from H to $H \otimes H^{op}$, the following can be defined:*

$$(\tau, W) \circ\!- (\rho, V) := (\tau \circ\!- \rho, V \multimap W),$$

where $\tau \circ\!- \rho$ is defined by

$${}^{(\tau \circ - \rho)(h)} f(-) := \sum \tau(h_1)(f(\rho(S(h_2))(-))).$$

${}^h f$ is an abbreviation for ${}^{(\tau \circ - \rho)(h)} f$ when ρ and τ are understood.

(exponential $\multimap$): *By using the fact that $(\mathtt{id} \otimes S^{-1}) \circ \Delta^{op}$ is an algebra map from H to $H \otimes H^{op}$, the following can be defined:*

$$(\rho, V) \multimap (\tau, W) := (\rho \multimap \tau, V \multimap W),$$

where $\rho \multimap \tau$ is defined by

$$f^{(\rho \multimap \tau)(h)}(-) := \sum \tau(h_2)(f(\rho(S^{-1}(h_1))(-))).$$

f^h is an abbreviation for $f^{(\rho \multimap \tau)(h)}$ when ρ and τ are understood.

(the unit for tensor product): *By virtue of the counit $\epsilon : H \to \mathbf{k}$, every vector space has the trivial representation of H, defined by $\rho(h)(v) := \epsilon(h)(v)$. We denote the trivial representation by (ϵ_V, V). Then $(\mathbf{1}, \mathbf{k})$ becomes the unit for tensor product with $\mathbf{1} := \epsilon_{\mathbf{k}}$.*

From the above, the dual representations are calculated as follows:
(the two dual representations $(-)^{\perp}$ and $^{\perp}(-)$):

$$^{\perp}(\rho, V) := (^{\perp}\rho, V^{\perp}), \qquad (\rho, V)^{\perp} := (\rho^{\perp}, V^{\perp}),$$

$$\textit{where } \ ^{\perp}\rho(h) f(-) = f(\rho(S(h)-)) \ \textit{ and } \ f^{\rho^{\perp}(h)}(-) = f(\rho(S^{-1}(h)-)).$$

Since a representation (ρ, V) is identified with an H-module V, Blute's $\mathcal{TMOD}(H)$ provides the two kinds of H-module structures on a homset $\mathcal{TVEC}(V, W)$ for given H-modules (V, ρ) and (W, τ): We call the H-module on $\mathcal{TVEC}(V, W)$ defined by $\rho \multimap \tau$ the *left* H-module and call that defined by $\tau \circ\!- \rho$ the *right* H-module. Then the following properties on $\mathcal{TVEC}$ with respect to left/right H-module structures are essential in our framework and assure that $(\mathcal{R})\mathcal{TMOD}(H)$ becomes a nonsymmetric (*)-autonomous category (cf. [6]), namely a *bi(-*-)autonomous category* (cf. [7, 10]) (Proposition 8 below):

Remark 3 *(i)Every natural isomorphism except for the symmetry isomorphisms for establishing the monoidal structure of $\mathcal{TVEC}$ is an isomorphism of H-modules; namely, $a_{U,V,W}$, r_V and l_V of Remark 2 of Section 2 are isomorphisms of H-modules for arbitrary H-modules whose underlying spaces are arbitrary objects U,V and W of $\mathcal{TVEC}$. (ii)Every natural isomorphism for establishing the autonomous structure of $\mathcal{TVEC}$ is an isomorphism either of right H-modules or of left H-modules; namely, the following ϕ_1 and ϕ_2 are isomorphisms of right H-modules and left H-modules respectively for arbitrary H-modules whose underlying spaces are arbitrary objects U,V and W of $\mathcal{TVEC}$:*

$$\begin{aligned} \phi_1 : \mathcal{TVEC}(U \otimes V, W) &\ \tilde{\to}\ \mathcal{TVEC}(U, \mathcal{TVEC}(V, W)), && (16)\\ \phi_2 : \mathcal{TVEC}(U \otimes V, W) &\ \tilde{\to}\ \mathcal{TVEC}(V, \mathcal{TVEC}(U, W)). && (17)\end{aligned}$$

Proof of Remark 3.
(i)For two objects V and V' of $\mathcal{TVEC}$, the morphism symmetry $s_{V,V'}$ of Remark 2 is not necessarily H-module isomorphism since $\Sigma h_1 \otimes h_2 \neq \Sigma h_2 \otimes h_1$. (ii) The following calculations assure that ϕ_2 is a left-H-module isomorphism with ϕ_2 defined by $(\phi_2(f)(v))(u) := f(u \otimes v)$.

$$\begin{aligned} (\phi_2(f)^h(v))(u) &= \Sigma(\phi_2(f)(S^{-1}(h_1) \cdot v))^{h_2}(u)\\ &= \Sigma h_3 \phi_2(f)(S^{-1}(h_1) \cdot v)(S^{-1}(h_2) \cdot u)\\ &= \Sigma h_3 f(S^{-1}(h_2) \cdot u \otimes S^{-1}(h_1) \cdot v).\end{aligned}$$

On the other hand,

$$\begin{aligned} (\phi_2(f^h)(v))(u) &= f^h(u \otimes v)\\ &= \Sigma h_2 \cdot f(S^{-1}(h_1) \cdot (u \otimes v))\\ &= \Sigma h_3 f(S^{-1}(h_2) \cdot u \otimes S^{-1}(h_1) \cdot v).\end{aligned}$$

Note that from the second line to the third line we use $\Delta \circ S^{-1} = (S^{-1} \otimes S^{-1}) \circ \Delta^{op}$. From the above $\phi_2(f)^h = \phi_2(f^h)$. The similar calculations assure that ϕ_1 is a right-H-module isomorphism with ϕ_1 defined by $(\phi_1(f)(u))(v) := f(u \otimes v)$.
□

From now on, for H-modules V and W, $V \multimap W$ and $W \circ\!- V$ denote the left and right H-module structures of $\mathcal{TVEC}(V, W)$ respectively. ${}^H\mathcal{TVEC}(V, W)$ and $\mathcal{TVEC}(V, W)^H$ denote $(V \multimap W)^H$ and $(W \circ\!- V)^H$ respectively, where U^H denotes the subspace $\{u \in U \mid h \cdot u = \epsilon(h)u\}$ of an H-module U. Then it is easily checked that

$$ {}^H\mathcal{TVEC}(U, V) = \quad (\mathcal{TMOD}(H))(U, V) = \quad \mathcal{TVEC}(U, V)^H. \tag{18} $$

Proposition 8 (Blute [7]) *$(\mathcal{R})\mathcal{TMOD}(H)$ becomes a bi(-*-)autonomous category.*

Proof.
From Remark 3 it is obvious that natural isomorphisms except for the symmetry isomorphisms for establishing the (*-)autonomous structure of $(\mathcal{R})\mathcal{TVEC}$ become natural isomorphisms for establishing the bi(-*-)autonomous structure of $(\mathcal{R})\mathcal{TMOD}(H)$; especially, if we restrict ϕ_1 and ϕ_2 of (16) and (17) to $\mathcal{TVEC}(U \otimes V, W)^H$ and ${}^H\mathcal{TVEC}(U \otimes V, W)$ respectively, then by (18) we obtain the following isomorphisms which establish the bi-autonomous structure of $\mathcal{TMOD}(H)$: Namely,

$$ \phi_1 : \mathcal{TMOD}(H)(U \otimes V, W) \quad \tilde{\to} \quad \mathcal{TMOD}(H)(U, W \circ\!- V). \tag{19} $$
$$ \phi_2 : \mathcal{TMOD}(H)(U \otimes V, W) \quad \tilde{\to} \quad \mathcal{TMOD}(H)(V, U \multimap W). \tag{20} $$

Moreover, ψ of Remark 2 of Section 2 establishes the dual structure of $\mathcal{RTMOD}(H)$:

$$ \psi : U \multimap V \quad \tilde{\to} \quad U^{\perp} \circ\!- V^{\perp}. $$

□

It is well known that a bi-*-autonomous category provides a model of noncommutative linear logic [10]. Blute's $\mathcal{RTMOD}(H)$ is considered as the first non trivial mathematical denotational model of the noncommutative linear logic. If one requires the condition that $V^{\perp} \tilde{\to} {}^{\perp}V$ for every object V in a bi-*-autonomous category, the resulting category called *cyclic *-autonomous category* (cf. [20]) becomes a model of Yetter's cyclic linear logic (cf. [25]). In the framework of $\mathcal{RTMOD}(H)$ Blute [7] showed the following:

Proposition 9 (Blute [7]) *If the antipode S is involutive (i.e., $S^2 = \mathtt{id}_H$), then $\mathcal{RTMOD}(H)$ becomes a cyclic-*-autonomous category.*

Proof. See the dual representations of the above Definition 11: $S^{-1} = S$ implies $(\rho, V)^{\perp} = {}^{\perp}(\rho, V)$.
□

A formula A of Cyclic Linear Logic is generated from atoms and constants of MLL by means of of the binary connectives of MLL augmented with $\circ\!-$ (cf. Appendix A for Cyclic Linear Logic). The interpretation $A^\bullet$ of A is defined by Definition 1 augmented with the interpretation of the binary connective $\circ\!-$ by its corresponding functor of $\mathcal{TVEC}$. With the same reason described in the paragraph below Definition 1, we often write $A(X_1^\bullet, \ldots, X_n^\bullet)$ for $A(X_1, \ldots, X_n)^\bullet$, where $X_1, \cdots, X_n$ is the list of atoms occurring in A. An interpretation of a sequent of the noncommutative linear logic by a homset in $\mathcal{TVEC}$ is defined by Definition 2. Then a proof π of a sequent $\Gamma \vdash \Delta$ is interpreted by the morphism $\pi^\bullet \in (\Gamma \vdash \Delta)^\bullet$. (See Definition 15 of Appendix B.) Then it is evident that the interpretation is preserved under the cut elimination procedure, namely, Proposition 1 also holds for $CyLL$. Now H-maps are defined analogously to Definition 6 with the terminology of $\mathcal{RTMOD}(H)$. These H-maps are proved to characterize the necessary and sufficient condition for a morphism to be an interpretation of a proof:

Definition 12 (H-invariant morphism:=H-map) *The same as Definition 6 in Section 2 except that G is read for H. $Hom_H((\Gamma \vdash \Delta)^\bullet)$ denotes the subspace of $(\Gamma \vdash \Delta)^\bullet$ consisting of the H-maps.*

For example, let $\pi_{A,B}$ denotes the unique cut free proof of $A \otimes B \vdash B \otimes A$ in MLL (note that the sequent is not provable in $CyLL$). Then $(\pi_{A,B})^{\bullet} = s_{A^{\bullet},B^{\bullet}}$ is not an H-map in general unless H is cocommutative.

Now the Soundness Theorem for $CyLL + Mix$ with respect to H-maps holds automatically:

Proposition 10 (soundness theorem of $CyLL + Mix$)
For any $CyLL + Mix$ proof π, $\pi^{\bullet}$ is an H-map.

Proof.
The same proof as that for Proposition 3 in Section 2 where G is read for H.
□

Blute-Scott [9] proved their completeness theorem by an elegant combinatorial method of representation of the shuffle Hopf algebra $\mathcal{A}$ (see [9] for the definition). Their proof of the most crucial Lemma 9.3 of [9] for obtaining their completeness theorem still works for our framework (Proposition 11 below) because their proof of the lemma does not assume dinaturality but $\mathcal{A}$-invariance.

Proposition 11 (Blute-Scott (cf. Lemma 9.3 of [9]))
For a binary sequent $\Gamma \vdash \Delta$ which is not cyclic (cf. Definition 5.3 of [9]),

$$Hom_{\mathcal{A}}((\Gamma \vdash \Delta)^{\bullet}) = \mathbf{0}.$$

Proof. See the Appendix C.
□

For a binary simple sequent $(\Gamma \vdash \Delta)^{\bullet}$ which is not provable in $MLL + Mix$, Corollary 4 of Section 3 implies $Hom_{\mathcal{A}}((\Gamma \vdash \Delta)^{\bullet}) = 0$ with the help of the following Proposition 12:

Proposition 12 *For a morphism ϕ in a homset which is an interpretation of a sequent, if ϕ is $\mathcal{A}$-invariant then ϕ is Z-invariant.*

Proof.
Given an arbitrary vector space V with an arbitrary representation ρ of Z, we define an $\mathcal{A}$-action by $x^n \cdot v := \rho(n)(v)/n! = \rho(1)(v)/(n-1)!$ (cf. pg.1432 of [9]), where $\mathcal{A}$ denotes the shuffle Hopf algebra $\mathbf{k}[\{x\}^*]$ (see [9] for the definition). We can directly check that $\mathcal{A}$-actions recover Z-actions; for example, for representations ρ and τ of Z, Z-actions on the tensor product is recovered by $(x^2 - x) \cdot (u \otimes v) = \rho(1)(u) \otimes \tau(1)(v)$, and Z-actions on the exponential is recovered by $f^{(x^2 - 6x^4)} = f^{(\rho \multimap \tau)(1)}$. Hence the assertion holds.
□

If a binary sequent is unprovable in $CyLL + Mix$ then it is not cyclic or unprovable in $MLL + Mix$ (cf. Lemma 5.4 of [9]). Hence with the help of Proposition 11 and 12, Corollary 4 (Section 3) with Z replaced by $\mathcal{A}$ is our completeness theorem for a binary simple sequent of $CyLL + Mix$. Finally with the help of Proposition 12, Theorem 1 (Section 3) with Z replaced by $\mathcal{A}$ is our unique interpretation theorem for a binary simple sequent of $CyLL + Mix$. Hence we have obtained the full completeness theorem for a binary sequent of $CyLL + Mix$, which is Proposition 6 where Z is read for $\mathcal{A}$. Finally we obtain the following full completeness theorem for an arbitrary sequent of $CyLL + Mix$:

Proposition 13 (full completeness theorem for an arbitrary sequent)
$\mathcal{ABS}_{\mathcal{A}}(M, N)$ has a basis consisting of interpretations of all distinct (modulo reductions for normalization) proofs of $M \vdash N$ in $CyLL + Mix$, where $\mathcal{ABS}_{\mathcal{A}}(M, N)$ is defined by the paragraph before Proposition 7 of Section 3 where Z is read for $\mathcal{A}$.

Analogously to our $MLL + Mix$ full completeness theorem, we can restate Proposition 13 as the following fully faithful representation theorem, where $\mathcal{F}_{cy}$ denotes the free cyclic *-autonomous category with Mix on a discrete graph of atoms and φ-$\mathcal{RTMOD}(\mathcal{A})$ denotes the same category defined before Proposition 7' where $\mathcal{RTVEC}$ is read for $\mathcal{RTMOD}(\mathcal{A})$.:

Proposition 13' (fully faithful representation theorem)
Every free functor $(\ )^\bullet : LIN(\mathcal{F}_{cy}) \to \varphi$-$\mathcal{RTMOD}(\mathcal{A})$ *is fully faithful (i.e., an equivalence) provided that* $(\ )^\bullet$ *is adequate.*

5 Discussion

Our full completeness theorem cannot apply if units are included. This is because units cannot be admitted in Danos-Regnier's sequentialization theorem [11], to which our full completeness theorem (as well as almost all the MLL full completeness theorems so far known) owes in the following manner; every morphism of a categorical model is attached to a proof structure, which is proved to be a proof net by the sequentialization theorem. It would be difficult to include units in a full completeness theorem of this nature.

The status of our restriction to binary sequents by means of the associated binary spaces, which were of rather syntactical flavor, has not been clear up to now. This is a delicate problem because there is no direct semantical counterpart of the restriction even though interpretation via dinatural transformations a la Blute-Scott [8] (rather than ours via morphisms) is considered. It would be a future work to clarify whether or not our interpretation provides the full completeness theorems for arbitrary nonbinary sequents without the restriction. The work would be important for clarifying the difference between our morphism interpretation and Blute-Scott's dinatural interpretation [8]. Among the differences a well known advantage of dinatural interpretation is that it provides the interpretation for the substitution rule, which is impossible via our morphism interpretation. For example, the similar advantage via natural transformation interpretation can be seen in Soloviev [23], where the coherence problem for symmetric monoidal closed category is reduced into that in the case of restricted sequents called 2-sequents.

For the pure syntax of MLL (without Mix), we [15] proved the full completeness theorem via dinatural interpretation in the *-autonomous category of reflexive topological abelian groups. One advantages of dinatural interpretation in [15] is that the corresponding theorem of Theorem 1 (unique interpretation theorem) can be reduced to the fact that the tensor unit (respectively, the dualizing object) is a generator (respectively, a cogenerator) for the *-autonomous category.

6 Appendix

6.1 Appendix A

In this subsection we shall define the syntax of the $MLL + Mix$ (Definition 13) and that of $CyLL + Mix$ (Definition 14).

Definition 13 (The $MLL + Mix$)

Initial sequents;

$$A \vdash A$$

$$\vdash \mathbf{1}$$

$$\perp \vdash$$

Rules;

$$\frac{\Gamma \vdash \Delta, A \quad A, \Lambda \vdash \Sigma}{\Gamma, \Lambda \vdash \Delta, \Sigma}\ cut$$

$$\frac{\Gamma \vdash \Delta, A \quad \Lambda \vdash B, \Sigma}{\Gamma, \Lambda \vdash \Delta, A \otimes B, \Sigma}\ \otimes right \qquad \frac{A, B, \Gamma \vdash \Delta}{A \otimes B, \Gamma \vdash \Delta}\ \otimes left$$

$$\frac{A, \Gamma \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}\ \multimap right \qquad \frac{\Gamma \vdash \Delta, A \quad B, \Lambda \vdash \Sigma}{\Gamma, A \multimap B, \Lambda \vdash \Delta, \Sigma}\ \multimap left$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}\ \perp right \qquad \frac{\Gamma \vdash \Delta}{\mathbf{1}, \Gamma \vdash \Delta}\ \mathbf{1} - left$$

$$\frac{\Gamma \vdash \Delta \quad \Lambda \vdash \Sigma}{\Gamma, \Lambda \vdash \Delta, \Sigma}\ mix$$

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ exchange - right \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ exchange - left$$

The $CyLL + Mix$ is defined as follows:

Definition 14 (The $CyLL + Mix$ (cf. [25]))
The CyLL + Mix is obtained from the MLL + Mix by deleting (exchange-right) and (exchange-left) and by adding the following rules:
Rules;

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, B \circ\!- A}\ \circ\!- right \qquad \frac{\Gamma \vdash A, \Delta \quad \Lambda, B \vdash \Sigma}{\Lambda, B \circ\!- A, \Gamma \vdash \Sigma, \Delta}\ \circ\!- left$$

$$\frac{\vdash A, B, \Delta}{\vdash B, \Delta, A}\ \textit{cyclic-exchange-right} \qquad \frac{\Gamma, A, B \vdash}{B, \Gamma, A \vdash}\ \textit{cyclic-exchange-left}$$

6.2 Appendix B

In this subsection we shall define the interpretation of a $CyLL+Mix$ proof ($MLL+Mix$ proof, simultaneously) by a morphism of the $*$-autonomous category $\mathcal{RTVEC}$.

Definition 15 (interpretation $\pi^\bullet$ of a proof π) *A proof π of $\Gamma \vdash \Delta$ for $CyLL+Mix$ (respectively, $MLL + Mix$) is interpreted by the morphism $\pi^\bullet$, where*

$$\pi^\bullet \in (\Gamma \vdash \Delta)^\bullet,$$

inductively as follows (respectively, in the following definition $\circ\!-$ is identified with $\multimap$);

(**initial sequents**) A proof π consisting of an initial sequent is interpreted by the following morphism $\pi^\bullet$;

$$\pi^\bullet := \mathtt{id}_{A^\bullet} \in (A \vdash A)^\bullet,$$

$$\pi^\bullet := \mathtt{id}_{\mathbf{k}} \in (\ \ \vdash \mathbf{1})^\bullet = (\perp \vdash \ \ )^\bullet.$$

For a sequence Γ of the formulas $A_1, \ldots, A_n$, ${}^{(\perp)}\Gamma^\perp$ denotes the sequence ${}^{(\perp)}A_n^\perp, \ldots, {}^{(\perp)}A_1^\perp$. $\acute{\Gamma}$ and $\grave{\Gamma}$ denote the sequences obtained from Γ by deleting the left most formula and right most formula, respectively. In the following $\Gamma^\bullet$ denotes $(\otimes\Gamma)^\bullet$ for a sequence Γ. Consider the interpretation $\pi^\bullet$ of a proof π of such a sequent of the form $\Gamma \vdash \Sigma, A, \Delta$ where Σ is not empty; $\pi^\bullet$ is an element from $Hom(\Gamma^\bullet \otimes (\Delta^\perp)^\bullet \otimes (A^\perp)^\bullet \otimes (\acute{\Sigma}^\perp)^\bullet, D^\bullet)$ with D denoting the left most formula of Σ. Because of the isomorphisms $(\mu_1)_V : V \tilde{\to} ({}^\perp V)^\perp$ and $(\mu_2)_V : V \tilde{\to} {}^\perp(V^\perp)$, there exists a canonical isomorphism in bi-*-autonomous category such that

$${}_\Sigma\delta_\Delta : Hom(\Gamma^\bullet \otimes (\Delta^\perp)^\bullet \otimes (A^\perp)^\bullet \otimes (\acute{\Sigma}^\perp)^\bullet, D^\bullet) \tilde{\to} Hom(({}^\perp\Sigma)^\bullet \otimes \Gamma^\bullet \otimes (\Delta^\perp)^\bullet, A^\bullet).$$

We define ${}_\Sigma\pi^\bullet_\Delta := {}_\Sigma\delta_\Delta(\pi^\bullet)$. ${}^\bullet\pi$ is written for ${}_\Sigma\pi^\bullet_\Delta$ when Δ is empty.

If a proof π comes from π_1 (and π_2 in the right hand side above the rule if exists) by the following rules, $\pi^\bullet$ is defined by means of $\pi_1^\bullet$ (and $\pi_2^\bullet$) as follows. In the following $A^\bullet$ denotes $\mathtt{id}_{A^\bullet}$ for a formula A. $\acute{C}$ and $\grave{C}$ denote the left most and the right most formulas of Δ, respectively. We shall use ϕ_1 of (19) and ϕ_2 of (20) and shall use canonical morphisms `rev` and `ev` which are the counits of ϕ_1 and of ϕ_2 respectively:

$$\mathtt{rev}_{V,W} : (W \circ\!- V) \otimes V \to W, \qquad\qquad \mathtt{ev}_{V,W} : V \otimes (V \multimap W) \to W.$$

Note that `rev` and `ev` are H-module morphisms from Remark 3.
(**cut**) ${}^\bullet\pi := {}^\bullet\pi_2 \circ (({}^\perp\grave{\Sigma})^\bullet \otimes {}^\bullet\pi_1 \otimes \Lambda^\bullet)$.
($\multimap$ **left**) ${}^\bullet\pi := {}^\bullet\pi_2 \circ (({}^\perp\grave{\Sigma})^\bullet \otimes \mathtt{ev} \circ ({}^\bullet\pi_1 \otimes (A^\bullet \multimap B^\bullet)) \otimes \Lambda^\bullet)$.
($\multimap$ **right**) $\pi^\bullet := \phi_2(\pi_1^\bullet)$.
($\otimes$ **left**) $\pi^\bullet := \pi_1^\bullet$ because $(A, B, \Gamma \vdash \Delta)^\bullet = (A \otimes B, \Gamma \vdash \Delta)^\bullet$.
($\otimes$ **right**) ${}_\Delta\pi^\bullet_\Lambda := {}^\bullet\pi_1 \otimes \pi_2^\bullet$.
(**mix**) ${}_\Delta\pi^\bullet_{\grave{\Sigma}} := \mathtt{ev} \circ (({}^\perp\grave{C})^\bullet \otimes \mathtt{mix} \circ ((\mu_1)_{\grave{C}^\bullet}({}^\bullet\pi_1) \otimes \pi_2^\bullet))$, where `mix` is defined in Corollary 3 of Section 3.
($\circ$**—right**) ${}^\bullet\pi := \phi_1({}^\bullet\pi_1)$.
($\circ$**—left**) $\pi^\bullet := \pi_2^\bullet \circ (\Lambda^\bullet \otimes \mathtt{rev} \circ ((B^\bullet \circ\!- A^\bullet) \otimes \pi_1^\bullet) \otimes (\grave{\Sigma}^\perp)^\bullet)$.
(**1-left**) $\pi^\bullet := \mathtt{u}_l \circ (\mathbf{1}^\bullet \otimes \pi_1^\bullet)$ with $\mathtt{u}_l : \mathbf{k} \otimes V \tilde{\to} V$.
($\perp$ **right**) $\pi^\bullet := \mathtt{ev} \circ (\pi_1^\bullet \otimes (\acute{C}^\perp)^\bullet)$ since $\perp^\bullet = \mathbf{k}$.
(**cyclic-exchange-right**) $\pi^\bullet := (\mu_2)^{-1}_{B^\bullet} \circ \phi_1(\mathtt{rev} \circ (\mathtt{rcy}_{A^\bullet} \otimes \pi_1^\bullet))$ with $\mathtt{rcy}_V : V^\perp \tilde{\to} {}^\perp V$.
(**cyclic-exchange-left**) $\pi^\bullet := \mathtt{ev} \circ (\mathtt{id}_{B^\bullet} \otimes (\mathtt{lcy}_{B^\bullet} \circ \phi_1(\pi_1^\bullet)))$ with $\mathtt{lcy}_V : {}^\perp V \tilde{\to} V^\perp$.
(**exchange-left**) $\pi^\bullet := \pi_1^\bullet \circ (\Gamma_1^\bullet \otimes \mathtt{s}_{A^\bullet, B^\bullet} \otimes \Gamma_2^\bullet \otimes (\acute{\Delta}^\perp)^\bullet)$ with $\mathtt{s}_{V,W} : V \otimes W \tilde{\to} W \otimes V$.
(**exchange-right**) If Δ_1 is empty, ${}_B\pi^\bullet_{\Delta_2} := \pi_1^\bullet \circ (\mathtt{s}_{(B^\perp)^\bullet, (\Gamma, \Delta_2^\perp)^\bullet}) \circ (\mathtt{lcy}_{B^\bullet} \otimes \Gamma^\bullet \otimes (\Delta_2^\perp)^\bullet)$. Otherwise the definition of $\pi^\bullet$ is the same as that of (exchange-left).

6.3 Appendix C

In this subsection, we shall demonstrate that Blute-Scott's proof for their Lemma 9.3 of [9] works as the proof for our Proposition 11 of Section 4. Before the demonstration of the proof, let us prepare the following proposition along the line of Blute-Scott [9].

Proposition 14 (Blute-Scott (cf. Lemma 9.1 of [9])) $\phi \in Hom_A(\mathbf{k}, A(X_1^\bullet, \cdots, X_n^\bullet))$ *is equivalent to the condition that for any representation* $\rho_1, \cdots, \rho_n$ *on* $X_1^\bullet, \cdots, X_n^\bullet$, *respectively,* $A(\rho_1^\bullet, \cdots, \rho_n^\bullet)(w) \circ \phi = \begin{cases} \phi & \textit{if } w = \epsilon \\ 0 & \textit{for any non empty word } w \end{cases}$.

Proof.
Since the unit $\mathbf{1}$ of $\mathcal{TMOD}(\mathcal{A})$ is determined by $\mathbf{1}(w)(k) = \begin{cases} k & \text{if } w \text{ is the empty word} \\ 0 & \text{for any non empty word } w \end{cases}$.
□

Proof of Proposition 11(cf. the proof of Lemma 9.3 of [9]).
There exist atoms X and Y such that either Y or $Y^{\perp}$ occurs between X and $X^{\perp}$ but its dual does not in the sequent $\vdash \Gamma^{\perp}, \Delta$. Now consider the Shuffle Hopf Algebra $\mathcal{A} := \mathbf{k}[\{x, y\}^{*}]$ and define the representations $(\rho, X^{\bullet})$ and $(\tau, Y^{\bullet})$ of $\mathcal{A}$ by $\rho(w)(a) := \begin{cases} \frac{a}{n!} & \text{if } w = x^n \\ 0 & \text{otherwise} \end{cases}$ and $\tau(w)(b) := \begin{cases} \frac{b}{n!} & \text{if } w = y^n \\ 0 & \text{otherwise} \end{cases}$, where w denotes a word generated by x and y. Let $B(X, Y, Z_1, \ldots, Z_n)$ denote $((\otimes\Gamma)\otimes(\otimes(\tilde{\Delta}^{\perp}))) \multimap C$, where $\Delta = C, \tilde{\Delta}$. Then for an arbitrary morphism $\phi : \mathbf{k} \to B(X^{\bullet}, Y^{\bullet}, Z_1^{\bullet}, \ldots, Z_n^{\bullet})$, $B(\rho, \tau, \epsilon_{Z_1^{\bullet}}, \ldots, \epsilon_{Z_n^{\bullet}})(xyx) \circ \phi = \phi$ holds (See [9] for detailed calculations of actions of words). This implies that $Hom_{\mathcal{A}}(\mathbf{k}, B(X^{\bullet}, Y^{\bullet}, Z_1^{\bullet}, \ldots, Z_n^{\bullet})) = 0$ by Proposition 14. From Remark 3, the assertion holds.
□

Acknowledgement: The author would like to express his sincere thanks to Prof. A. Scedrov for his criticism and suggestions on an earlier version of this paper. The author also would like to express his sincere thanks to Prof. R.F. Blute and Prof. P.J. Scott who suggested the author a possibility that his results of Section 2 and 3 can be naturally extended to a noncommutative case.

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