Softness of MALL Proof-Structures and a Correctness Criterion with Mix

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Abstract

We show that every MALL proof-structure [9] satisfies the property of *softness*, originally a categorical notion introduced by Joyal. Furthermore, we show that the notion of *hereditary softness* precisely captures Girard's algebraic restriction of *the technical condition* on proof-structures. Relying on this characterization, we prove a MALL+Mix sequentialization theorem by a proof-theoretical method, using Girard's notion of *jump*.

Our MALL+Mix correctness criterion subsumes the Danos/Fleury-Retoré criterion [6] for MLL+Mix.

1 Introduction

One of the primary contributions of Linear Logic is that it has provided a graph-theoretic treatment of proofs. This is done by giving a *correctness criterion*, a graph-theoretic characterization, which distinguishes genuine proofs among general proof-structures. Technically proof-structures are abstract graphs consisting of formulas/links. *Correct* proof-structures satisfying this criterion are called *proof-nets.* This program has been particularly successful in Multiplicative Linear Logic (MLL): Several beautiful criteria have been introduced, including Girard's *long trip* condition [7] and Danos-Regnier's *connected/acyclic* condition [4].

By extending the graph theoretical mechanism to additive connectives, Girard [9] introduced MALL proof-structures and obtained a sequentialization theorem among them. His proof-structures are an extension of MLL proof-structures, obtained by attaching boolean algebras (called weights) to each formula/link occurrence. From the very beginning of linear logic [7], a notion of proof-structure which included additives was proposed using *additive-boxes*, which are of a rather syntactic flavor. The MALL structures in [9] are more abstract and free from syntax in that they omit the use of additive-boxes and instead introduce the abstract algebraic notion of boolean weight. Girard's ingenious idea for the sequentialization theorem is to introduce a notion of *jump*, by which a larger class of graphs is obtained (as opposed to only allowing slices). The result of Girard [9] is a correctness criterion for MALL, which subsumes Danos-Regnier's criterion: A proof-structure Θ is sequentializable for MALL if and only if for all switchings S, the graph Θ_S is connected and acyclic.

Although Girard's MALL proof-structures are genuinely graph theoretical, Girard requires a rather difficult algebraic restriction, referred to as *the technical condition*. Our first purpose in this paper is to give an intuitive proof-theoretical understanding of this condition. We do this by establishing a notion of *softness* on the proof-structures. Softness is originally a categorical notion, introduced by Joyal [13], which concerns a factorization property between products and coproducts. The property is especially important when considering products and coproducts (the categorical counterpart of the additives) over a *-autonomous category (the categorical model for MLL). Behind its category theoretical formalization, the notion has a direct logical/syntactical analogue, which can be thought as partial invertibility of the $\oplus$-rule. Specifically, the property states that if $\vdash A_{11} \oplus A_{12}, \ldots, A_{n1} \oplus A_{n2}$ is MALL provable then $\exists i \in \{1, \ldots, n\}\, \exists k \in \{1, 2\}$ such that $\vdash A_{11} \oplus A_{12}, \ldots, A_{ik}, \ldots, A_{n1} \oplus A_{n2}$ is MALL provable. This syntactical analogue is a weak form of Andreoli's focalisation [2].

In this paper, a proof-structure Θ (which has other links besides axiom-links) is defined to be *soft* if it has a terminal link, i.e., a link L such that the conclusion of L becomes a conclusion of Θ and the boolean weight attached to L is 1. This notion of softness for proof-structures could be regarded as a (slightly generalized) counterpart of Joyal's categorical notion of softness in the following sense: given a proof structure Θ whose conclusions are $A_{11} \oplus A_{12}, \ldots, A_{n1} \oplus A_{n2}$, an arbitrary terminal link L of Θ (if it exists) must be a $\oplus_j$-link (cf. Definition 20) with $j \in \{1, 2\}$. Hence softness for such a Θ means that Θ has a terminal $\oplus_j$-link. Equivalently we can say that it has a removable $\oplus_j$-link by virtue of every terminal $\oplus_j$-link being removable. This is exactly a counterpart of Joyal's softness via the straightforward correspondence between "removal of a $\oplus_j$-link" and "factorization through a coproduct $\oplus$". The notion of softness of proof-structures is naturally extended to a more dynamic notion of *hereditary softness* in accordance with focalisation.

We begin by showing that every MALL proof-structure is soft (Theorem 1). This is a rather surprising consequence of Girard's algebraic restriction on proof-structures. Conversely we show that the extended notion of *hereditary softness* implies Girard's technical condition (Corollary 3). This is proved via a sequentialization theorem for hereditarily soft PALL proof-structures (Proposition 2), where PALL is the fragment of MALL without $\otimes$, i.e., the allowable connectives are $\parr$, & and $\oplus$. Hence we are able to conclude that Girard's technical condition is equivalent to the hereditary softness of proof structures (Theorem 2).

The second purpose of this paper is to give a MALL+Mix sequentialization theorem among MALL proof-structures. We shall prove that a MALL proof-structure is sequentializable for MALL+Mix if and only if for every switching S the graph Θ_S is acyclic (Theorem 3). This subsumes the Danos/Fleury-Retoré criterion for MLL+Mix [6]. Our proof of the theorem is crucially based on our direct characterization of MALL proof-structures by means of softness. In the case of MLL it is now well known that if we add the Mix rule, the MLL correctness criteria are generalized into Asperti's deadlock freeness [3] and into Danos/Fleury-Retoré acyclicity condition [6]. Among the difficulties in proving the multiplicative sequentialization theorem with Mix, there is the fact that the original notion of *empire* can no longer be used. The situation is the same in the MALL case, since the proof in [9] relies on this same idea. Moreover, a peculiar difficulty intrinsic to additives is that a wider class of jumps is necessary to obtain a criterion for Mix (cf. Section 5.1): Under the constraint of the connectedness condition of Girard's original MALL criterion, local jumps are sufficient for the criterion (where we say that a jump is *local* if the target and the origin of the jump is in a same connected component in a slice). On the other hand, in the presence of Mix, which removes the connectedness constraint, local jumps are not sufficient; i.e., a jump can go from one connected component to another in a slice. Thus, the class of jumps required for our criterion is wider than for the original MALL criterion. A MALL sequentialization theorem in the presence of Mix is therefore by no means trivial.

Our proof requires a careful analysis of Girard's jumps in MALL+Mix sequentializable proof-structures, although our targets are genuine MALL proof-structures which are not necessarily sequentializable. The reason is described by the following slogans (i) and (ii):

(i) Every proof-structure is recovered from a sequentializable proof-structure: Let PALL+Mix be the fragment of MALL+Mix without $\otimes$. Softness of MALL proof-structures implies that every MALL proof-structure is recoverable from a certain PALL+Mix sequentializable proof-structure by replacing some $⅋$-links (together with inherited occurrences of $⅋$ in descendant formulas) by $\otimes$-links (together with occurrences of $\otimes$) (cf. Corollary 2).

(ii) All jumps are inherited from those in clusters: To investigate weight assignments (particularly possible jumps) for MALL proof-structures, it is crucial to analyze those restricted for the class of MALL+Mix sequentializable proof-structures, because weights of a proof-structure are preserved under renaming of $⅋$-links with $\otimes$-links together with (i). We introduce a special class of MALL+Mix sequentializable proof-structures, called *clusters*, from which every MALL+Mix sequentializable proof-structure is constructed without any considerable constraint of weights (cf. Definition 12 and 14). We show that our clusters geometrically well behave like axiom-links (Proposition 3 and Corollary 4).

Through the above investigations, our main technical lemma in this paper is a splitting lemma (Proposition 4) for *simple* proof-structures (Definition 16). Our proof for the lemma is given constructively by a proof-theoretical method. Sequentializable information on $|\Theta|$, which is an important byproduct of softness of proof-structures, plays a role. Based our work on the splitting lemma, our MALL+Mix sequentialization is carried out in (Theorem 3).

The paper is organized as follows: in Section 2 we first prove that all Girard's MALL proof-structure are soft (Theorem 1). We also introduce the notion of hereditary softness and prove that it exactly captures Girard's technical condition on structures (Theorem 2). In Section 3 we first give a detailed algorithm for demonstrating the adequacy of MALL+Mix sequentializable proof-structures. Second we introduce clusters, which form a special class of sequentializable proof-structures, and prove certain geometric properties of clusters (Proposition 3 and Corollary 4). In Section 4 we prove the splitting lemma (Proposition 4) for simple proof-structures by using Corollary 4. Using this splitting lemma, we prove our MALL+Mix sequentialization theorem (Theorem 3). In Section 5 we remark that a wider class of cycles is necessary for our MALL+Mix criterion than for MALL. We also remark on the dependency condition in our sequentialization proof. In an appendix, for the reader's convenience, we give the definition of MALL proof-structures.

2 Girard's MALL Proof-Structures Are Soft

In this section we first show that every MALL proof-structure [9] satisfies a *softness* condition, a notion originating in work of Joyal [13]. Our goal ultimately is to show that *hereditary softness* exactly characterizes Girard's algebraic restriction of the technical condition on proof-structures. For the definition of MALL proof-structure (p-s in short), see Definition 20 of the appendix.

We shall start with the definition of softness of MALL proof-structure. For this we remind the reader of the definition of a terminal link:

Definition 1 (terminal link (cf. Definition 4 of [9])) A link L which is not an axiom-link in a proof-structure Θ is called *terminal* if a conclusion of L is in turn a conclusion of Θ and $w(L) = 1$. Note that every terminal $\{⅋, \&, \oplus_1, \oplus_2\}$-link can be removed (see Definition 4 of [9] for the removal of terminal links).

In what follows, $\oplus_i$-link denotes either a $\oplus_1$-link or a $\oplus_2$-link.

Definition 2 (softness for proof-structure) A proof-structure Θ is called *soft* if Θ has a terminal link or Θ has no other link except axiom-links.

Here we remind the reader of Joyal's categorical notion of softness:

Definition 3 (Joyal's softness [13, 10]) A $*$-autonomous category with products and coproducts, which is a categorical model for MALL, is called *soft* if every morphism f of type $\vdash A_{11} \oplus A_{12}, \ldots, A_{n1} \oplus A_{n2}$ (with an arbitrary natural number n) factors through some k ($1 \leq k \leq n$) via a coproduct injection into $A_{k1} \oplus A_{k2}$.

A beautiful example of soft category is Ehrhard's category of hypercoherences [5], which is a refinement of Girard's category of coherent spaces [8]. Ehrhard obtained softness by considering hypergraphs rather than graphs on a web.

Our notion of softness of proof-structures could be seen as a (slightly generalized) counterpart of Joyal's categorical notion of softness: Consider the case where every link (except an axiom) whose conclusion is in turn a conclusion of Θ is either a $\oplus_1$-link or $\oplus_2$-link. Softness of Θ means that Θ has a terminal link, which in this case is a $\oplus_i$-link with $i \in \{1, 2\}$. Since an arbitrary terminal $\oplus_i$-link can be removed (cf. Remark 1 below), softness of such Θ means that there exists a (terminal) $\oplus_i$-link which can be removed. This is exactly a counterpart of Joyal's softness via the straightforward correspondence between "removal of a $\oplus_i$-link" and "factorization through coproduct $\oplus_i$" .

Example 1
1. Every M$\oplus$LL proof-structure is soft, where a M$\oplus$LL proof-structure is a p-s without any &-link.
2. Every MALL sequentializable proof-structure is soft.

The following is a key proposition to obtain softness of MALL proof-structures (Theorem 1). To obtain it, Girard's technical condition (cf. Definition 20 of the appendix) is crucial. The following proof, which works without the assumption of monomiality of boolean weights, is essentially due to Dominic Hughes [11]. Be careful: we however implicitly use the monomiality since the formalization of technical condition in our paper is equivalent to Girard's original one only for monomial weights:

Proposition 1 *If Θ is a* MALL *proof-structure with &-links, then Θ has a &-link L such that $w(L) = 1$.*

Proof Suppose for deriving a contradiction that for every &-link L in Θ, $w(L) \neq 1$, hence $w(L)$ depends on some eigenweight. Choose an arbitrary &-link, say L_i, in Θ, then $w(L_i)$ depends on some eigenweight, say p_{i+1}, associated with some &-link, say L_{i+1}. From the technical condition (cf. (i') of Definition 20 in Appendix), we have $w(L_i) \subset w(L_{i+1})$. Hence starting with $i = 1$, we have sequence of &-links L_i whose weights satisfy:

$$w(L_1) \subset w(L_2) \subset \cdots \subset w(L_i) \subset w(L_{i+1}) \subset \cdots$$

Since &-links in Θ are finite, $\exists i \, \exists j \; (j < i + 1$ and L_{i+1} is $L_j)$. Thus we have

$$w(L_{i+1}) = w(L_j) \subset \cdots \subset w(L_i) \subset w(L_{i+1})$$

This is a contradiction. $\square$

A natural but non-trivial observation is the following:

Theorem 1 (MALL proof-structures are soft)
Every MALL *proof-structure* Θ *is soft.*

Proof If Θ is a M$\oplus$LL proof-structure, it is obviously soft. Hence we may assume that Θ contains at least one &-link. By Proposition 1, there exists in Θ a &-link L such that $w(L) = 1$. Hereditarily below L there is a link K whose conclusion, say A, is in turn a conclusion of Θ; i.e., A is a hereditary conclusion of L. Then it holds that $w(K) \supseteq w(L) = 1$. Thus we have $w(K) = 1$, hence K is a terminal link. $\square$

Discussion 1 (Joyal's softness is a weak form of Andreoli's focalisation) The proof-theoretical property of Joyal's softness naturally occurs in the well known framework of Andreoli's focalisation [2]. The framework is well established to capture a fundamental proof-theoretical property of linear logic. The property is described in terms of removing of connectives: the negative connectives $\{\parr, \&\}$ are *invertible*: i.e., if $\vdash \Gamma, A_1 \parr A_2$ and $\vdash \Gamma, A_1 \& A_2$ are **MALL** provable, then so are $\vdash \Gamma, A_1, A_2$ and $\vdash \Gamma, A_i\ \forall i \in \{1, 2\}$, respectively. The dual property of the invertibility is *focalisation* for the positive connectives $\{\otimes, \oplus\}$: i.e., whenever positive connectives can be removed, they can be done simultaneously until we reach negative (sub)formulas. Note that focalisation for positive connectives does not ensure any determinism in removing connectives but ensures synchronization of several removals at once. Although positive connectives are far from invertible in general, there is one exceptional case; one may encounter, after removing all negative connectives, a special positive sequent which has no outer-most $\otimes$; i.e., the sequent must be of the form $\vdash \ell_1, \ldots, \ell_k, A_{11} \oplus A_{12}, \ldots, A_{n1} \oplus A_{n2}$ with ℓ_i's denoting literals. Then focalisation guarantees that one can non-deterministically, remove some $\oplus$ from the sequent. This is exactly what Joyal's softness says. Hence Joyal's softness can be regarded as a weak form of Andreoli's focalisation in the sense that a special case of the latter is the former.

What follows is devoted to showing that Girard's algebraic restriction of the technical conditions corresponds to our *hereditary softness* of structures (cf. Definition 6), which is a proof theoretic property rather than an algebraic one. The notion fits very naturally with focalisation as described in Discussion 2. Our goal is to show an equivalence between Girard's technical condition and our notion of hereditary softness.

We start with

Definition 4 (quasi-proof-structure) A *quasi-proof-structure* (*q-p-s* in short) Ξ is a proof-structure without Girard's technical condition.

Definition 5 (mapping $|\ \ |$ among quasi-proof-structures) For a quasi-proof-structure Ξ, $|\Xi|$ denotes the quasi-proof-structure obtained from Ξ by replacing all the $\otimes$-links (together with all occurrences of $\otimes$) by $\parr$-links ($\parr$ occurrences).

It is straightforward to observe:

Ξ satisfies the technical condition iff $|\Xi|$ satisfies the technical condition.

Hence

$$\Xi \text{ is a proof-structure} \quad \text{iff} \quad |\Xi| \text{ is a proof-structure.}$$

The removal of a terminal link (cf. Definition 4 of [9]) can be defined not only for proof-structures but also for quasi-proof-structures.

Remark 1 Every terminal $\{\parr, \&, \oplus_1, \oplus_2\}$-link of a quasi-proof-structure can be removed.

In the following, PALL denotes MALL without $\otimes$, i.e., the allowable connectives are $\parr$, $\&$ and $\oplus$.

From the above remark if Ξ is a PALL q-p-s (i.e., a q-p-s without $\otimes$-links), by removing a terminal link of Ξ we obtain a new q-p-s $\tilde{\Xi}$. Then we have the following definition. Note that the notion of softness is defined not only for proof-structures but also for quasi-proof-structures.

Definition 6 (hereditary softness) A PALL q-p-s Ξ is called *hereditarily soft* if either of the following holds:

- All the links of Ξ are axiom-links.
- Ξ is soft and every proof-structure $\tilde{\Xi}$ obtained by removing an arbitrary terminal link of Ξ is *hereditarily soft*.

Discussion 2 (Hereditary softness is focalisation for PALL) Softness is a rather static notion on a fixed proof-structure. Hereditary softness is an extension of this static notion into a dynamic one. It is a property on series of proof-structures arising successively through splittings of terminal links. Accordingly, hereditary softness is defined in terms of removing terminal links. This is reminiscent of Andreoli's focalisation, which has been defined in terms of removal of logical connectives. Indeed hereditary softness can be regarded as focalisation for the restricted fragment PALL: note first that every negative terminal $\{\parr, \&\}$-link can be removed (cf. Definition 1). Thus hereditary softness says that, unless Θ consists only of axiom-links, every proof-structure Θ obtained after removing all negative terminal links remains soft. Hence it contains a positive terminal link, which must be a $\oplus_i$-link in PALL . Note second that every terminal $\oplus_i$-link can be removed, as opposed to the other positive $\otimes$-links, which however do not occur in PALL . Note third that removing of $\oplus_i$-link preserves all weights of links remained. This is quite opposite to the case of the other additive $\&$-links, whose removal involves valuating an associated eigenweight, hence changes weights. Hence while removing successive terminal $\oplus_i$-links starting from Θ, no terminal negative link arises until our removing reaches a negative link hereditarily above the first removed $\oplus_i$. Hence once we choose a terminal $\oplus_i$-link of Θ, we can synchronously remove all $\oplus_i$-links successively occurring above the link until reaching a first negative link above. This is exactly focalisation in the absence of $\otimes$. Indeed this discussion yields Proposition 2 below, where we prove PALL+Mix sequentialization for all p-s's without $\otimes$ under the assumption of hereditary softness.

As a corollary of Theorem 1 we have:

Corollary 1 *Every* PALL *proof-structure* Θ *is hereditarily soft.*

Proof By induction on the number of links except axiom-links in Θ.
(Base Case:) All the links in Θ are axiom-links, thus Θ is hereditarily soft.
(Induction Case:) Since Θ is soft by Theorem 1, in this case Θ has a terminal link, which is either $\parr$, $\&$, $\oplus_1$ or $\oplus_2$-link. By Remark 1 the link can be removed to obtain a PALL proof-structure, which is hereditarily soft by I.H. Hence Θ is hereditarily soft. $\square$

It is straightforward to extend the definition of *sequentializable* proof-structure (Definition 5 of [9]) in the presence of Mix:

Definition 7 (MALL+Mix sequentializable proof-structures) A proof-structure Θ is called MALL+Mix *sequentializable* when either of the following holds:

- Θ is an axiom-link.
- Θ has a terminal link which can be removed and the removal yields sequentializable proof-structures.
- Θ is a union of two proof-structures which are sequentializable.

Every MALL+Mix sequentializable proof-structure is associated with a MALL+Mix proof. The converse is the adequacy theorem (Lemma 3) shown in the next section.

The following is a key proposition to obtain Corollary 3 below:

Proposition 2 (Hereditary softness implies PALL+Mix sequentialization) *Suppose Ξ is a hereditarily soft PALL q-p-s. Then Ξ is sequentializable for PALL+Mix.*

Proof By induction on the number of links except axiom-links in Ξ.
(Base Case:) All the links of Ξ are axiom-links, hence Ξ is sequentializable for Mix.
(Induction Case:) Since Ξ is particularly soft, in this case Ξ has a terminal link which is either a $⅋$, $\&$, $\oplus_1$ or $\oplus_2$-link. By Remark 1 and hereditary softness of Ξ, the link can be removed to obtain q-p-s's, which are hereditarily soft, thus by I.H., are sequentializable. Hence Ξ is sequentializable. $\Box$

As is clear from this proof, Proposition 2 is a witness of Discussion 2 above.

From Corollary 1, we have

Corollary 2 *Every PALL proof-structure is sequentializable for PALL+Mix. Hence for every MALL proof-structure Θ, $|\Theta|$ is sequentializable for PALL+Mix.*

We have another corollary (Corollary 3) of Proposition 2. For the corollary the following lemma is crucial.

Lemma 1 *If a quasi-proof-structure is MALL+Mix sequentializable then it satisfies Girard's technical condition.*

Proof We can check the assertion straightforwardly by using the algorithm shown in the adequacy theorem (Lemma 2 below). $\Box$

Corollary 3 (Hereditary softness implies Girard's technical condition) *Suppose Ξ is a MALL q-p-s such that $|\Xi|$ is hereditarily soft. Then Ξ satisfies Girard's technical condition, hence is a proof-structure.*

Proof By Proposition 2, $|\Xi|$ is PALL+Mix sequentializable. By the above lemma $|\Xi|$ satisfies the technical condition, hence so does Ξ. $\Box$

Combining Corollary 1 and Corollary 3, we have the following goal of this section:

Theorem 2 (Equivalence of hereditary softness and technical condition) *The following are equivalent for an arbitrary quasi-proof-structure Ξ:*

1. *Ξ satisfies Girard's technical condition, (alternatively Ξ is a proof-structure).*
2. *$|\Xi|$ is hereditarily soft.*

3 Geometric Properties of Clusters

In this section first of all we shall prove the adequacy theorem for MALL. We shall do this by giving the algorithm for interpreting MALL proofs into MALL sequentializable proof-structures. The adequacy theorem for MALL+Mix is straightforward from that for MALL. Second we shall introduce a geometrically well-behaved class of MALL+Mix sequentializable proof-structures, called *clusters* (Definition 12). We shall introduce a *clusters-representation* which provides a representation of every MALL+Mix sequentializable p-s from clusters without the considerable constraint of weights (Definition 14). Our goal in this section is to derive certain important geometric properties of clusters (Proposition 3 and Corollary 4).

All proof-structures considered in this section are MALL+Mix sequentializable.

3.1 The Algorithm of Additive Adequacy Theorem

This subsection is devoted to proving the adequacy theorems for MALL and for MALL+Mix. The proof will be algorithmic, and this algorithm will later be used in obtaining Proposition 3. The latter will be crucially used in the splitting lemma in the next section.

Lemma 2 (Adequacy theorem for MALL) *Every MALL proof π is interpreted by a sequentializable MALL proof-structure Θ_π. Note that Θ_π is not necessarily unique in the presence of the &-rule, as opposed to the situation in MLL.*

Proof
We prove the lemma by induction on the length of the proof π. We only consider the crucial case where π is obtained from π_1 and π_2 by application of the &-rule (other cases are straightforward):

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash C_1,\ldots,C_n,A\end{array}\quad \begin{array}{c}\vdots\ \pi_2\\ \vdash C_1,\ldots,C_n,B\end{array}}{\vdash C_1,\ldots,C_n,A\&B}\ (\&)$$

Before the proof we introduce some necessary terminologies:

Definition 8 (common part) *The common part* $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ consists of occurrences of formulas/links with weights as follows:
(i) All the C_i are occurrences of $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ with the original weight 1.
(ii) Suppose a formula X is an occurrence in $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$. If all the links whose conclusion is X in Θ_{π_1} coincide (including weights) with those in Θ_{π_2}, then the links occur in $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ with the original weights.
(iii) Suppose L is a link occurrence in $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$. The premises of L (if they exist), which in turn are common both Θ_{π_1} and Θ_{π_2}, occur in $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ with the original weights.

The bottom-most occurrences of $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ are the occurrences $C_1,\ldots,C_n$. $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ can be seen to be the largest sub-occurrences which are common to both those in Θ_{π_1} and in Θ_{π_2} and which lie above (and including) $C_1,\ldots,C_n$. When we say that the common part is common to both Θ_{π_1} and Θ_{π_2}, we mean this to include the weight assignments. By virtue of (iii), any topmost occurrence in $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ is either a formula or an axiom-link.

Definition 9 (boundary) A *boundary* $\partial(\Theta_{\pi_1},\Theta_{\pi_2})$ is a list $D_1,\ldots,D_k$ of occurrences of formulas and axiom-links in $\nabla(\Theta_{\pi_1}\Theta_{\pi_2})$ such that for every topmost occurrence X of $\nabla(\Theta_{\pi_1}\Theta_{\pi_2})$, there exists exactly one D_j which is hereditarily below X or equal to X. For example, the list of the occurrences $C_1,\ldots,C_n$ is a boundary and the list of topmost occurrences in $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ is a boundary.

Definition 10 (territory) For the common part $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ and a (fixed) boundary $\partial(\Theta_{\pi_1},\Theta_{\pi_2})$, the *territory* $\partial\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ is the set of occurrences resulting from $\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ by erasing every occurrence (of formula/link) which lies hereditarily above some D_i of the boundary $\partial(\Theta_{\pi_1},\Theta_{\pi_2})$. Hence, the topmost occurrences of $\partial\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ are $D_1,\ldots,D_k$ which are the boundary $\partial(\Theta_{\pi_1},\Theta_{\pi_2})$. For an arbitrary boundary, the bottom-most occurrences of a territory $\partial\nabla(\Theta_{\pi_1},\Theta_{\pi_2})$ are the occurrences $C_1,\ldots,C_n$.

Now we begin the proof: By the induction hypothesis, π_1 and π_2 are interpreted by proof-structures Θ_{π_1} and Θ_{π_2}. Θ_π is constructed from Θ_{π_1} and Θ_{π_2} as follows after choosing a boundary $\partial(\Theta_{\pi_1}, \Theta_{\pi_2})$ [1]:

In the following, for an occurrence Y of a formula/link in Θ, $\Theta(Y)$ denotes the sub-occurrences of Θ consisting of Y and all the occurrences hereditarily above Y in Θ. One constructs Θ_π by the following steps.

- One begins by introducing a unique link L whose conclusion is the conclusion $A\&B$ of a proof-structure Θ_π, We define $w(L) = 1$. An associated eigenweight p for the &-link L is introduced. For the two premises of L, we define $w(A) = p$ and $w(B) = \neg p$.
- One includes the territory $\partial\nabla(\Theta_{\pi_1}, \Theta_{\pi_2})$ as a part of the bottom of Θ_π. Every axiom link (if exists) in a given boundary $\partial(\Theta_{\pi_1}, \Theta_{\pi_2})$ in turn becomes a topmost occurrence in Θ_π.
- For a formula occurrence D in $\partial(\Theta_{\pi_1}, \Theta_{\pi_2})$, let $L_{11}, \ldots, L_{1n_1}$ (respectively $L_{21}, \ldots, L_{2n_2}$) denote the list of all links whose conclusion is the occurrence D in Θ_{π_1} (respectively in Θ_{π_2}). The copies $L'_{11}, \ldots, L'_{1n_1}, L'_{21}, \ldots, L'_{1n_2}$ of these links are defined to be the links whose conclusion is the occurrence D in Θ_π. These $(n_1 + n_2)$-links in Θ_π are defined to be all links whose conclusion is the occurrence D. We define $w(L'_{1k}) = p.w(L_{1k})$ and $w(L'_{2k}) = \neg p.w(L_{2k})$. This definition satisfies the disjointness property of Definition 20 (iii): i.e., $w(L'_{11}), \ldots, w(L'_{1n_1}), w(L'_{21}), \ldots, w(L'_{2n_2})$ are pairwise disjoint since so are $w(L_{11}), \ldots, w(L_{1n_1})$ and $w(L_{21}), \ldots, w(L_{2n_2})$.
- $\Theta_\pi(L'_{1k})$ is obtained from $\Theta_{\pi_1}(L_{1k})$ by changing each weight w into $p.w$. $(1 \leq k \leq n_1)$
- $\Theta_\pi(L'_{2k})$ is obtained from $\Theta_{\pi_2}(L_{2k})$ by changing each weight w into $\neg p.w$. $(1 \leq k \leq n_2)$
- $\Theta_\pi(A)$ is obtained from $\Theta_{\pi_1}(A)$ by changing each weight w into $p.w$.
- $\Theta_\pi(B)$ is obtained from $\Theta_{\pi_2}(B)$ by changing each weight w into $\neg p.w$.

□

We remark the following on the algorithm for the interpretation of the &-rule, which has been described above in our proof of Lemma 2:

Remark 2 Let π be a MALL proof obtained from MALL proofs π_1 and π_2 by an application of &-rule. Θ_π denotes a MALL proof-structure obtained from Θ_{π_1} and Θ_{π_2} as in Lemma 2 and p denotes the eigenweight associated with the &-link interpreting the last &-rule of π.

(i) Every occurrence in Θ_{π_1} and in Θ_{π_2} has a (unique) image in Θ_π. Moreover if a weight of an occurrence in Θ depends on p, then the inverse image of the occurrence is a singleton, hence is in one of Θ_{π_1} and Θ_{π_2}.

(ii) In an arbitrary associated proof-structure Θ_π, the following are all occurrences whose weights depend on p; (1) all occurrences in $\Theta_\pi(A)$ (2) all occurrences in $\Theta_\pi(B)$ (3) all occurrences hereditarily above the boundary $\partial(\Theta_{\pi_1}, \Theta_{\pi_2})$.

Example 2 (proof-structure Θ_π for a proof π) We give some examples of the algorithm described in the adequacy theorem:

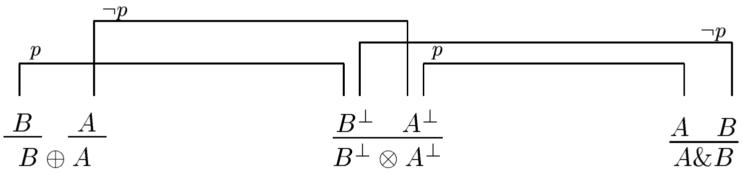
[1] Θ_π is not uniquely determined from Θ_{π_1} and Θ_{π_2} since a choice $\partial(\Theta_{\pi_1}, \Theta_{\pi_2})$ of a boundary is not unique. This is why an interpretation of a MALL proof into a MALL (sequentializable) proof-structure is not unique, as was mentioned in Remark 3 after Definition 5 of Girard [9]

- Consider the following proof π whose last inference is &.

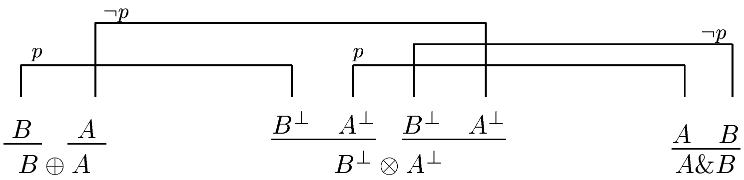
$$\dfrac{\dfrac{\dfrac{\vdash B, B^\perp \quad \vdash A^\perp, A}{\vdash B, B^\perp \otimes A^\perp, A}}{\vdash B \oplus A, B^\perp \otimes A^\perp, A} \quad \dfrac{\dfrac{\dfrac{\vdash B, B^\perp \quad \vdash A^\perp, A}{\vdash B, B^\perp \otimes A^\perp, A}}{\vdash A, B^\perp \otimes A^\perp, B}}{\vdash B \oplus A, B^\perp \otimes A^\perp, B}}{\vdash B \oplus A, B^\perp \otimes A^\perp, A\& B}\ \&$$

Let π_1 and π_2 denote the subproofs whose conclusions are the left and right premises of the &-inference, respectively. According to two possible choices of a boundary $\partial(\Theta_{\pi_1}, \Theta_{\pi_2})$, we obtain two kinds of interpretations Θ_π described in (1) and (2):

(1)

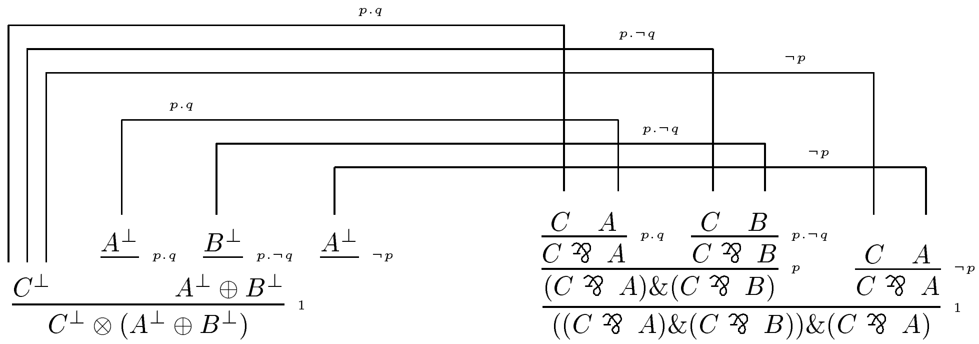


(2)



- The following proof-structure Θ_π is one possible interpretation for the following MALL proof π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash C^\perp, C \quad \dfrac{\vdash A^\perp, A}{\vdash A^\perp \oplus B^\perp, A}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), C, A}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), C \parr A} \quad \dfrac{\dfrac{\vdash C^\perp, C \quad \dfrac{\vdash B^\perp, B}{\vdash A^\perp \oplus B^\perp, B}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), C, B}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), C \parr B}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), ((C \parr A)\&(C \parr B))} \quad \dfrac{\dfrac{\vdash C^\perp, C \quad \dfrac{\vdash A^\perp, A}{\vdash A^\perp \oplus B^\perp, A}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), C, A}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), C \parr A}}{\vdash C^\perp \otimes (A^\perp \oplus B^\perp), ((C \parr A)\&(C \parr B))\&(C \parr A)}$$



It is straightforward to extend our adequacy theorem into MALL+Mix while preserving the property of the above Remark 2:

Lemma 3 (Adequacy theorem for MALL+Mix) *Every* MALL+Mix *proof* π *is interpreted by a* MALL *proof-structure* Θ_π *which is sequentializable for* MALL+Mix.

Definition 11 (splitting of a proof-structure) A splitting of a proof-structure is either of

1. A removal of a terminal link of Θ.
2. A division of Θ into two proof-structures whose disjoint union is Θ.

1 corresponds to a MALL rule and 2 corresponds to a Mix rule.

Remark 3 (splitting is closed in sequentializable p-s's) If Θ is a MALL+Mix sequentializable p-s, then p-s's obtained after a splitting of Θ are still MALL+Mix sequentializable. In particular, if Θ has a terminal &-link L, it can always be removed (cf. Remark 1). Thus we denote by $\Theta[p_L = 1]$ and by $\Theta[p_L = 0]$ the two MALL+Mix sequentializable p-s's after removing L, where p_L is an eigenweight for L: i.e., $\Theta[p_L = 1]$ (respectively $\Theta[p_L = 0]$) denotes the sequentializable p-s resulting from Θ by replacing p to 1 (respectively to 0) and by removing formulas/links of weights 0.

In what follows "sequentializable" means "MALL+Mix sequentializable".

3.2 Clusters-Representation for a Sequentializable P-S and Geometric Properties of Clusters

Definition 12 (cluster) A *cluster* T is a MALL+Mix sequentializable p-s interpreting a proof π such that the followings $(i) \sim (iv)$ are satisfied:

(i) The last inference of π is & or axiom; i.e.,

$$\pi \text{ is} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots \\ \vdash \Delta, C_1 & \vdash \Delta, C_2\end{array}}{\vdash \Delta, C_1 \& C_2}\,(\&) \quad \text{or} \quad \vdash \alpha, \alpha^\perp.$$

(ii) T has no terminal $\{\parr, \oplus_1, \oplus_2\}$-link.
(iii) No terminal $\otimes$-link in T can be removed.
(iv) T cannot be a disjoint union of proof-structures.

Note that if in (i), T is an interpretation of an axiom, then $(ii) \sim (iv)$ are redundant. A cluster is called an *axiom-cluster* or *&-cluster* according to the last inference of π. To stress π, the above cluster is denoted by T_π. Note that for a &-cluster T_π there may be another proof π' such that $T_{\pi'} = T_\pi$. In particular the principal formulas of the last inferences of π and π' may be different. Since any terminal &-link can be removed and by Remark 3 removal of a terminal link is closed in the class of sequentializable p-s's, the clause above (i) is alternatively given by the following (i');

(i') T is a MALL+Mix sequentializable p-s which has a terminal &-link or which consists only of one axiom-link (together with its conclusions).

Remark 4 From a given p-s satisfying (i) above (equivalently (i')), a set of clusters is obtained by successive applications of the following procedures:

- To remove all terminal $\{\parr, \oplus_1, \oplus_2\}$-links and all removable terminal $\otimes$-links.
- To divide disjoint union of two proof-structures.

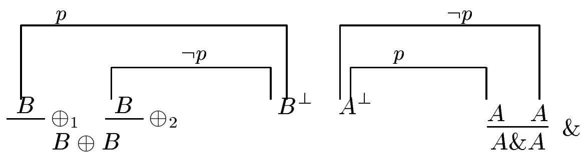
Thus a cluster can be seen as a prime sequentializable p-s satisfying (i) (equivalently (i')).

This remark yields the following alternative definitions of cluster:

Remark 5 (alternative definitions of cluster) The followings are equivalent:

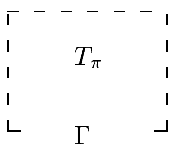
- T is a cluster.
- T is a MALL+Mix sequentializable proof-structure such that T is an axiom link (together with its conclusion) or every possible sequentialization for T starts with a removal of a terminal &-link.
- T is a MALL+Mix sequentializable proof-structure such that it is impossible to remove any terminal link of T except &-links and to divide T into two disjoint proof-structures. That is, every possible splitting of T is a removal of a (terminal) &-link.

Example 3 The following MALL+Mix sequentializable p-s is a cluster. Be careful: it cannot be a disjoint union of proof-structures since the left half graph (consisting of formulas occurrences $B^{\perp}, B, B, B \oplus B$ together with the links whose conclusions are the formulas) is no longer a proof-structure.



Definition 13 (principal links for a cluster) Let T denote a cluster. For a fixed π such that $T = T_\pi$, the unique link corresponding to the last inference of π is called the *principal link of* T *with respect to* π. A link L is called a *principal link* for T if L becomes the principal link of $T_\pi = T$ with respect to some proof π. Every principal link is a terminal link. By noting the alternative (i') in Definition 12, it is equivalently defined by saying that a *principal link* is a terminal &-link or a terminal axiom-link. A principal link for a &-cluster T is not unique in general, as mentioned below Definition 12. A *principal formula* for T is a conclusion of a principal link for T. Obviously every principal formula is a conclusion of a cluster.

Throughout the paper we use a doted square to represent a cluster T of conclusions Γ as follows:



The following is an important property for clusters on replacement of $\otimes$ by $\parr$ so as to preserve being a cluster.

Lemma 4 (replacing a non-terminal $\otimes$-link with $\parr$-link in a cluster T) *For a non-terminal $\otimes$-link L of a cluster T, a proof-structure T' is obtained from T by replacing L (together with inherited occurrences of $\otimes$ in descendant formulas) with a $\parr$-link (together with occurrences of $\parr$). Then T' remains a cluster.*

Proof First, T' has a sequentialization, which is that of T by replacing the corresponding $\otimes$-removal of L by a $\parr$-removal, together with a Mix splitting to follow immediately. Second, since the replaced link is not terminal, every removable link for T' has the same removal in the original T. This means that every sequentialization of T' must start with a splitting of a &-link, hence T' is a cluster. $\square$

Of course the lemma fails for a replacement of a terminal $\otimes$-link by a $\parr$-link, which violates the condition (ii) of Definition 12 of cluster. The lemma will be used in Preliminary Reduction 2 in the proof of Proposition 3 below.

We have the following definition by noting Remark 4:

Definition 14 (clusters-representation for a sequentializable p-s)
Every MALL+Mix sequentializable p-s can be represented from union of clusters $T_1, \dots, T_n$ by drawing $\{\otimes, \parr, \oplus_1, \oplus_2\}$-links of weights 1 below them such that iterated removals of these links yield the union of clusters. That is, the p-s is obtained from $T_1, \dots, T_n$ by (sound) applications of MALL+Mix rules without any use of &-rules. We say this representation is a *clusters-representation* for a sequentializable p-s and call each T_i a *component* of the clusters-representation. In particular if a given p-s is an MLL proof-structure, then the components $T_1, \dots, T_n$ are all the axiom-links (together with these conclusions) in the p-s. Those links/formulas which are not contained in any components constitute the *end-part* of a representation. Note that every link whose conclusion in turn is a conclusion of a p-s is terminal if the link occurs in the end-part.

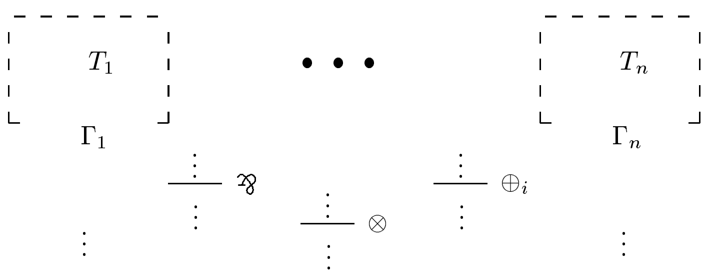


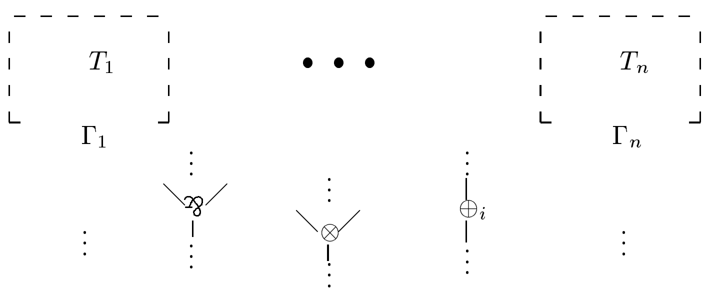
Figure 1: A clusters-representation for a sequentializable p-s

Remark 6 (end-part in terms of removing links) All links in the end-part can be removed successively before removing any links in clusters so that the removals yield the union of clusters. Hence in particular, they can be removed without any preceding removal of a &-link.

Remark 7 (unique link property in the end-part) For every formula X in the end-part, there exists a unique link whose conclusion is X.

In terms of MALL proof-structure presentation a la Laurent, this remark states that the end-part contains no *additive contraction node* (cf. [14]).

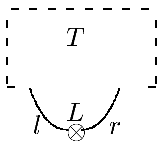
Given the above remark, throughout the paper the proof-structure in Figure 1 can be illustrated as follows so that the links in the end-part are represented by edges, which makes formula occurrences in the end-part implicit:



We say a link in the end-part is *hereditarily below a cluster* if the link is hereditary below some conclusion of the cluster. We say *a cluster is hereditary above a link in the end-part* if the link is hereditarily below the cluster.

The following remark, which is a direct implication of Remark 6, is a crucial property of a $\otimes$-link in the end-part:

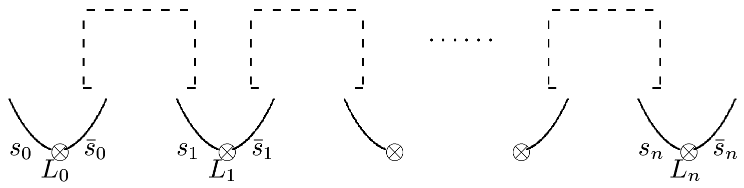
Remark 8 ($\otimes$-links in the end-part and clusters) For a $\otimes$-link L in the end-part, let $Clu(L, l)$ (respectively $Clu(L, r)$) denote a set of clusters which have a conclusion C hereditarily above L such that the *left* (respectively *right*) premise of L has an inherited occurrence of C. Obviously the set of all clusters hereditarily above L is a union of $Clu(L, l)$ and $Clu(L, r)$. Suppose for contradiction that the two sets contain a common cluster T. Then inherited occurrences of two different conclusions of T appear respectively in left and right premises of L, as is illustrated by the following:



Hence this contradicts Remark 6 that removal of links in the end-part yields the union of clusters. Thus Remark 6 directly implies the following;

$$Clu(L, s) \cap Clu(L, \bar{s}) = \emptyset, \text{ where } \{s, \bar{s}\} = \{l, r\}.$$

We generalize this remark more generally to a series $L_0, L_1, \ldots, L_n$ of different terminal $\otimes$-links in the end-part, which satisfies $Clu(L_{m-1}, \bar{s}_{m-1}) \cap Clu(L_m, s_m) \neq \emptyset$ for all m less than n, where $\{s_m, \bar{s}_m\} = \{l, r\}$ for each $m = 0, 1, \ldots, n$ (see the following figure);



Then by the same reasoning as above, Remark 6 directly implies the following;

$$Clu(L_0, s_0) \cap Clu(L_n, \bar{s}_n) = \emptyset.$$

Next we prepare the following lemma on the clusters-representation for $T[p = i]$ with $i \in \{1, 0\}$, where T is a &-cluster and p is an associated eigenweight with a principal link for T (cf. Remark 3 for the notation).

Throughout this subsection $w(\ )$ always denotes the weights-function for a whole proof-structure T. Let us start with the following lemma:

Lemma 5 (fundamental lemma on a clusters-representation of $T[p = i]$) *Let T be a &-cluster and p denote the associated eigenweight with a principal link for T. Let us consider a clusters-representation consisting of components $U_1, \ldots, U_n$ for $T[p = i]$ with a fixed $i \in \{1, 0\}$. Then following holds for each U_j:*

- *Suppose no $w(M)$ depends on p for all links M in U_j. Then there is a $\otimes$-link L hereditarily below U_j. Moreover suppose there exists a terminal $\otimes$-link L' one of whose (direct) premises is a conclusion of U_j. Then there exists a $\otimes$-link L different from L' hereditarily below U_j.*

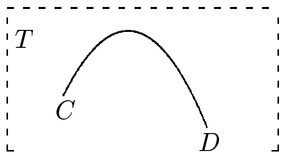
Proof Suppose no weights $w(M)$ of links M in U_j depend on p. This in particular implies that every conclusion of U_j has weight 1 in T, hence so does every link hereditarily below U_j in T. Thus every link (if it exists) hereditarily below a conclusion of U_j must appear in the end-part of $T[p = i]$, hence must be a $\{\otimes, \oplus_1, \oplus_2, \parr\}$-link. Thus if there exists a link hereditarily below a conclusion of U_j, so does a terminal $\otimes$-link since otherwise there arises a terminal $\{\oplus_1, \oplus_2, \parr\}$-link, which contradicts the condition (ii) of Definition 12 for a cluster T. Now suppose for contradiction that there exists no $\otimes$-link hereditarily below U_j. Then from the above, there exists no link hereditarily below U_j; that is, every conclusion of U_j is in turn a conclusion of T. By the first supposition on the weights of links in U_j, T is divided into two proof-structures, one of which is U_j. This contradicts the condition (iv) of Definition 12 for a cluster T.

Moreover suppose the existence of a $\otimes$-link L' as given in the assertion. Suppose for contradiction that L' is the only $\otimes$-link hereditarily below U_j. This means by the first supposition that the $\otimes$-link L can be removed. This contradicts the condition (iii) of Definition 12 for a cluster T. $\square$

The following is a main proposition in this subsection. The purpose of the proposition is to show that our clusters behave geometrically like axiom-links, which are the base case of the proposition:

In the following for formula occurrences X and Y in a sequentializable proof-structure, we denote $X \sim_S Y$ if there exists a switching S to yield a path between X and Y. We often write $X \sim Y$ by omitting S.

Proposition 3 (main proposition on a cluster) *For any formula occurrence C of a cluster T there exists a principal formula D and a switching S such that $C \sim_S D$. D is chosen to be a different occurrence from C unless T has the unique principal formula which happens to be C itself.*



Proof Let $n(T)$ denote the sum of the number of links in every slice of T. We shall prove the assertion by triple induction on

$$m(T) := (n(T), n_+(T), n_-(T)),$$

where $n_+(T)$ is the number of $\otimes$-links of weights 1 of T and $n_-(T)$ is the number of $\parr$-links of weights 1 of T.

(Base Case) This is a case where T is an axiom-cluster. Hence the proposition is obvious.

(Induction Case) In this case T is a &-cluster.
First we shall introduce Preliminary Reductions $1 \sim 3$, which satisfy the following properties:

- The reductions are closed in the class of clusters and make the value of $m(\ )$ strictly decrease.
- The assertion of the proposition for a reduced p-s implies that for an original p-s (the converse does not hold in general in Preliminary Reduction 2).

(Preliminary Reduction 1) when T has a $\oplus_i$-link L of weight 1:
L can be eliminated by skipping it to obtain a proof-structure T' whose conclusions

are obtained from those of T by replacing the corresponding conclusion $A_1 \oplus A_2$ of L by a certain A_i. Note that T' is still a cluster and $n(T') < n(T)$. It is straightforward to observe that the assertion for T' implies that for T.

(Preliminary Reduction 2) when T has a $\otimes$-link of weight 1:
Let L denote a non terminal $\otimes$-link of weight 1. By replacing L with a $\parr$-link as described in Lemma 4, a proof-structure T' is obtained. By Lemma 4, T' is a cluster and we have $n_+(T') < n_+(T)$ together with $n(T') = n(T)$. It is immediate to observe that the assertion for T' implies that of T, since every graph arising through the $\parr$-link of T' in turn becomes a subgraph through the replaced $\otimes$-link of T.

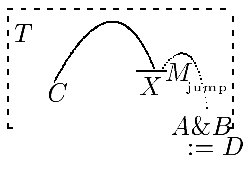
(Preliminary Reduction 3) when T has a &-link L of weight 1 whose conclusion in turn is a premise of a $\parr$-link:
To L we can apply a p-s counterpart of the distribution of & over $\parr$, which we shall define in the paragraph before Lemma 6, so as to obtain a proof-structure $\widehat{T}$. By Lemma 6 (i), $\widehat{T}$ is a cluster. The application directly yields that $n_-(\widehat{T}) < n_-(T)$ together with $n_+(\widehat{T}) \leq n_+(T)$ and $n(\widehat{T}) = n(T)$. It is important to observe that by Lemma 6 (iii), the assertion for $\widehat{T}$ implies that for T.

We may assume that there exists a principal &-link (for T) whose conclusion is not the given occurrence C; otherwise T has a unique principal link whose conclusion happens to be C. In the latter case, the proposition is obvious. Hence from now on let $A\&B$ and p denote respectively the conclusion and the associated eigenweight of a principal &-link whose conclusion is not the given occurrence C. Since a principal link is a terminal &-link, the sequentializable p-s's $T[p = 1]$ and $T[p = 0]$ are obtained by evaluating p to 1 and 0, respectively. In the following proof, $w(\ )$ denotes the weight-function for T.

From now on, suppose for contradiction that *the assertion of the proposition does not hold for some cluster T with some formula occurrence C* ; i.e., for any principal formula D of T and any switching S it is impossible to have $C \sim_S D$. The supposition means that the set, say H, consisting of clusters for which the assertion does not hold is non-empty. First we have the following straightforward claim for a cluster $T \in H$:

(Preliminary Claim:) No formula lying on every path $C \sim_{S_i} X$ starting from C in T is a conclusion of a link whose weight depends on p.

<u>Proof of Claim:</u> Suppose for contradiction that there exists a path $C \sim X$ such that X is a conclusion of a link M s.t. $w(M)$ depends on p. Then X can be chosen as a jump with respect to the &-link with conclusion $A\&B$. Then $C \sim X \sim A\&B$, thus the assertion for T holds by taking D in the proposition to be $A\&B$. See the following figure.



<u>End of Proof for Claim</u>

In the non-empty H, there exists a cluster T such that $m(T)$ is minimal. Hence for every cluster U such that $m(U) < m(T)$, the assertion for U must hold, which we shall refer as induction hypothesis (I.H. in short). From now on, in what follows, we fix a *minimal* $T \in H$ together with a formula occurrence C in H.

Note that the above Preliminary Reductions $1 \sim 3$ are closed in the set H; i.e., if a p-s is in H then a reduced p-s is in H (cf. the note before preliminary reductions). Thus a minimal $T \in H$ must not have any $\{\otimes, \oplus_1, \oplus_2\}$-link of weight 1 and must not have any &-link of weight 1 whose conclusion is a premise of a $⅋$ -link.

By virtue of Preliminary Claim, in particular $w(C)$ does not depend on p. Hence C appears both in $T[p=1]$ and in $T[p=0]$. We consider a clusters-representation for the sequentializable p-s $T[p=1]$, whose components we denote by $U_1, \ldots, U_n$. From now on, until the end of the proof, a clusters-representation always denotes that for $T[p=1]$. Note that every formula occurrence in the end-part for the representation is a hereditary conclusion of a terminal link of some component U_i. Hence we may assume that C appears in some component U_i (with $i \in \{1, \ldots, n\}$) of clusters-representation for $T[p=1]$.

Step 0:
In this step the following cases arise for the formula C:

(Case 1)$_0$ The case where C is the unique principal formula of U_i:
In this case, first we have the following claim:

(Claim 1)$_0$ U_i has no link whose weight depends on p.

Proof of Claim: The proof is same to the proof of (Claim 1)$_m$ below in general (Step m) when m is read by 0 (i.e., $U_{i_0} := U_i$ and $C_0 := C$). End of Proof for Claim

Lemma 5 together with this claim implies that in this (Case 1)$_0$, there exists a $\otimes$-link hereditarily below U_i. Let E_0 denote a U_i's conclusion such that hereditarily below E_0 there exists a $\otimes$-link $L(E_0)$ in the end-part of $T[p=1]$. By a certain switching in the end-part, we have $E_0 \sim L(E_0)$. Since $n(U_i) < n(T)$ and C is the only principal formula in U_i, by I.H. on a cluster U_i with the occurrence E_0, there exists a switching S_i for U_i such that $E_0 \sim_{S_i} C$. We have $C \sim L(E_0)$ by noting that switchings for a component and for the end-part are independent.

(Case 2)$_0$ Negation of (Case 1)$_0$: The case where C is not a unique principal formula of U_i.
Since # (&-links in U_i) < # (&-links in T), I.H. on C in a cluster U_i implies that there exists in U_i a principal formula E_0 and a switching S_i for U_i such that $C \sim_{S_i} E_0$ and E_0 and C are different occurrences. Let L_0 denotes the principal link of conclusion E_0 in U_i. Since L_0 is terminal in U_i, the result from $w(L_0)$ by evaluating p to 1 is 1. Hence we have either that $w(L_0)$ depends on p or that $w(L_0) = 1$. However by virtue of Preliminary Claim above, the former does not hold, hence $w(L_0) = 1$. Then we have the following straightforward claim:

(Claim 2)$_0$: L_0 does not remain terminal in T.

Proof of Claim: The proof is the same to the proof of (Claim 2)$_m$ below in general (Step m) when m is 0 (i.e., $U_{i_0} := U_i$ and $C_0 := C$). End of Proof for Claim

By virtue of this claim, in this (Case 2)$_0$, there exists a $\otimes$-link $L(E_0)$ hereditarily below E_0 in the end-part of $T[p=1]$ since all links below $L(E_0)$, which are $\{\otimes, ⅋, \oplus_1, \oplus_2\}$-links, have weights 1 and the condition (ii) of Definition 12 holds for a cluster T. By a certain switching in the end-part, we have $E_0 \sim L(E_0)$. We have $C \sim L(E_0)$ by noting that switchings for a component and for the end-part are independent.

To sum up for (Step 0), in either cases (Case 1)$_0$ and (Case 2)$_0$, we have a path $C \sim_{S_i} E_0 \sim L(E_0)$ for a conclusion E_0 of U_i and a terminal $\otimes$-link $L(E_0)$ hereditarily below E_0 in the end-part. Note that the path appears independently of a valuation of p by Preliminary Claim. Moreover by virtue of Remark 8, there must exist hereditarily above $L(E_0)$ a cluster U_{i_1} with $i \neq i_1$. See Figure 3 (After Step m) at the bottom of the proof where m is read by 0 (i.e., $U_{i_0} := U_i$ and $C_0 := C$).

Generally, the following Step m *must* occur inductively for $m \geq 1$: Note that each (Case X)$_m$ introduced below together with (Claim X)$_m$ corresponds to the above (Case X)$_0$ together with (Claim X)$_0$, where $X \in \{1, 2\}$.

Step m for $m \geq 1$:
In this step, to a terminal $\otimes$-link $L(E_{m-1})$ arising in the preceding Step $m-1$, the following (Case 1)$_m$ and (Case 2)$_m$ will arise. Note that until the preceding steps, we have inductively the following (cf. Figure 2 (Before Step m)):
$C \sim_{S_i} E_0 \sim L(E_0) \sim C_1 \sim_{S_{i_1}} E_1 \sim L(E_1) \sim \cdots \sim C_{m-1} \sim_{S_{i_{m-1}}} E_{m-1} \sim L(E_{m-1})$.
Here C_k and E_k are different conclusions of a cluster U_{i_k} for $1 \leq k \leq m-1$ and $U_i, U_{i_1}, \ldots, U_{i_{m-1}}$ are different clusters and $L(E_0), L(E_1), \ldots, L(E_{m-1})$ are different $\otimes$-links by (Claim $m-1$) in step $m-1$ for $m > 1$. This implies that $C \sim L(E_{m-1})$ by remarking that switchings for different components and for the end-part are independent. Moreover by virtue of Remark 8, there must exist hereditarily above $L(E_{m-1})$ a cluster U_{i_m} such that $i_m \notin \{i, i_1, i_2, \ldots, i_{m-1}\}$. Let us fix arbitrarily some U_{i_m}'s conclusion C_m which is hereditarily above $L(E_{m-1})$ (cf. Figure 2).

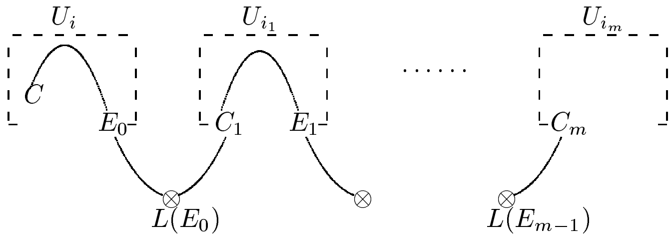


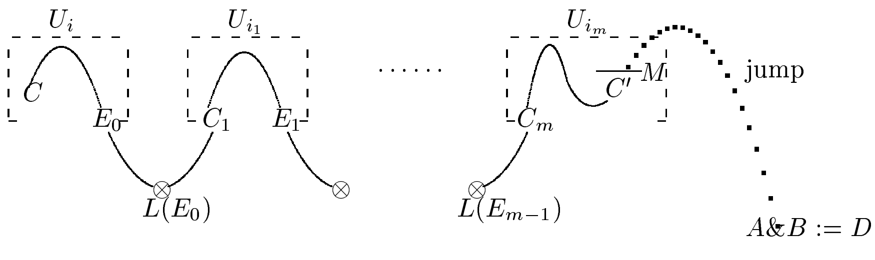
Figure 2: Before Step m

(Case 1)$_m$ The case where C_m is the unique principal formula of U_{i_m}.
In this case, note that C_m is a &-formula such that $w(C_m) = 1$ by virtue of Preliminary Claim. Hence by virtue of Preliminary Reductions 1,2 and 3, C_m must be a (direct) premise of the $\otimes$-link $L(E_{m-1})$.

In this case first we have the following claim:

(Claim 1)$_m$ U_{i_m} has no link whose weight depends on p.

<u>Proof of Claim:</u> Suppose for contradiction that U_{i_m} has a link M whose weight $w(M)$ depends on p. Let C' denote a conclusion of M. Since $n(U_{i_m}) < n(T)$ and C_m is the only principal formula in U_{i_m}, by I.H. on a cluster U_{i_m} with the occurrence C', there exists a switching S_{i_m} for U_{i_m} such that $C' \sim_{S_{i_m}} C_m$. By choosing C' as a jump with respect to the &-link of conclusion $A\&B$, we have $A\&B \sim C' \sim_{S_{i_m}} C_m$, thus we have shown the assertion for T by taking D of the proposition to be $A\&B$. See the following figure. This is a contradiction.



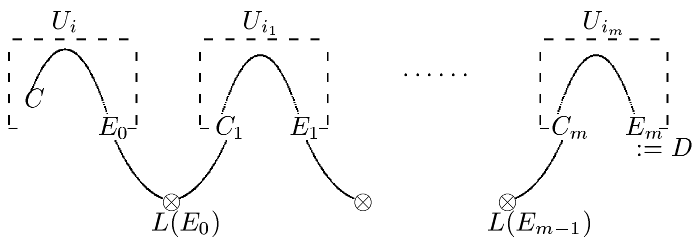
<u>End of Proof for Claim</u>

By virtue of this claim and of the note below (Case $1)_m$, Lemma 5 implies that there exists a conclusion E_m of U_{i_m} such that E_m is a different occurrence from C_m and that there is a terminal $\otimes$-link, say $L(E_m)$, hereditarily below E_m. By a certain switching in the end-part, we have $E_m \sim L(E_m)$. Since $n(U_{i_m}) < n(T)$ and C_m is the only principal formula in U_{i_m}, by I.H. on a cluster U_{i_m} with the occurrence E_m, there exists a switching S_{i_m} for U_{i_m} such that $E_m \sim_{S_{i_m}} C_m$. Hence we have $C_m \sim L(E_m)$ by noting that switchings for a component and for the end-part are independent. (cf. Figure 3 at the bottom of the proof).

(Case 2)$_m$ Negation of (Case $1)_m$: The case where C_m is not the unique principal formula of U_{i_m}.
Since # (&-links in U_{i_m}) < # (&-links in T), I.H. on C_m in a cluster U_{i_m} implies that there exists in U_{i_m} a principal formula E_m and a switching S_{i_m} for U_{i_m} such that $C_m \sim_{S_{i_m}} E_m$ and E_m and C_m are different occurrences. Let L_m denote the principal link with conclusion E_m in U_{i_m}. Since L_m is terminal in U_{i_m}, the result from $w(L_m)$ by evaluating p to 1 is 1. Hence we have either that $w(L_m)$ depends on p or that $w(L_m) = 1$. However by virtue of Preliminary Claim above, the former does not hold, hence $w(L_m) = 1$. Then we have the following straightforward claim:

(Claim $2)_m$ L_m does not remain terminal in T.

<u>Proof of Claim:</u> If L_m remains terminal in T, then the assertion for T holds by taking D in the proposition to be E_m. See the following figure.



<u>End of Proof for Claim</u>

By virtue of this claim, in this case (Claim $2)_m$ there exists a $\otimes$-link $L(E_m)$ hereditarily below E_m in the end-part of $T[p = 1]$ since all links below L_m, which are $\{\otimes, \parr, \oplus_1, \oplus_2\}$-links, have weights 1 and the condition (ii) of Definition 12 holds for a cluster T.

In either cases (Case $1)_m$ and (Case $2)_m$ for (Step m), we have a path $C_m \sim_{S_{i_m}} E_m \sim L(E_m)$ for a conclusion E_m of U_{i_m} and a terminal $\otimes$-link $L(E_m)$ hereditarily below E_m in the end-part (cf. Figure 3 at the bottom of the proof). Note that by Preliminary Claim, the path appears independently of a valuation of p. Now the terminal $\otimes$-link $L(E_m)$ which so arises satisfies the following claim, which is crucial for Step m with $m \geq 1$:

(Claim m)

The $\otimes$-link $L(E_m)$ is different from the $\otimes$-link $L(E_i)$ for every $i \in \{0, 1, \ldots, m-1\}$.

<u>Proof of Claim:</u> Suppose for contradiction that $\exists k \in \{0, 1, \ldots, m-1\}$ such that $L(E_k)$ is $L(E_m)$. The supposition means that there exists a binary link M hereditarily above $L(E_k)$ and hereditarily both below C_{k+1} and E_m since the two different occurrences E_m and C_{k+1} are hereditarily above $L(E_k)$. Since every link N hereditarily below E_m is in the end-part of $T[p = 1]$, it must not be a &-link. Moreover

for $L(E_m)$, N is not terminal and has weight 1, by virtue of Preliminary Reductions 1 and 2, it must be a $⅋$-link. Thus in particular M must be a $⅋$-link (see the following figure for T). The proof is divided into the following two cases (a) and (b).

(a) For $L(E_m)$ arising in (Case 2)$_m$ such that U_{i_m} is a &-cluster:
In this case E_m is a conclusion of a &-link of weight 1. By virtue of Preliminary Reduction 3 together with what we have just said on N above, E_m must be a direct premise of $L(E_m)$. However this contradicts the existence of the $⅋$-link M.

(b) Negation of (a): That is, for $L(E_m)$ arbitrarily arising in (Case 1)$_m$ and for $L(E_m)$ arising in (Case 2)$_m$ such that U_{i_m} is an axiom-cluster:
It is important to note first that in this case E_m is not a conclusion of a principal &-link for a cluster U_{i_m}. From the construction from Step k to Step m, we have the following path

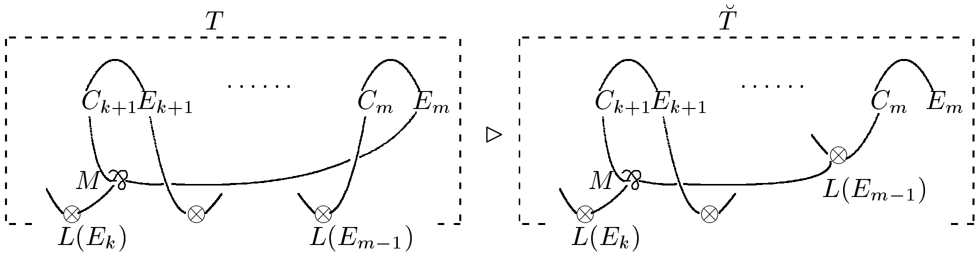
$$L(E_k) \sim C_{k+1} \sim E_{k+1} \sim \cdots \sim L(E_{m-1}) \sim C_m \sim E_m. \tag{1}$$

Since the path (1) is a continuation of the path $C \sim L(E_k)$, as has been shown in Figure 2, no formula lying on (1) is a conclusion of a link whose weight depends on p by Preliminary Claim. This means that the path (1) appears independently of valuations of p. By changing the $⅋$-switching for the $⅋$-link M, we have the following path

$$C_{k+1} \sim E_{k+1} \sim \cdots \sim L(E_{m-1}) \sim C_m \sim E_m \sim L(E_k). \tag{2}$$

Since the path between E_m and $L(E_k)$ arises through links whose weights are 1 via the subformula-relation, the path (2) also appears independently of valuations of p (cf. the following figure T for paths (1) and (2)). In the presence of the paths (1) and (2) in certain graphs arising in T, throughout a sequentialization of T, it is impossible to remove any $L(E_j)$ with $k+1 \leq j \leq m-1$ before M is removed. On the other hand M cannot be removed before $L(E_k)$ is removed. Thus we remark (†) Every removal of $L(E_k)$ must proceed those of $L(E_j)$ with $k+1 \leq j \leq m-1$ throughout a sequentialization of T.

Now we shall transform T into a well defined proof-structure $\breve{T}$ such that $m(T) = m(\breve{T})$ by the following procedure: For a unique link whose premise is E_m in T, replace the premise E_m by the conclusion of $L(E_{m-1})$. This yields successive replacements of the inherited occurrences of E_m by the $L(E_{m-1})$'s conclusion and makes E_m terminal. Thus we obtain a proof-structure $\breve{T}$ (see the following figure for $\breve{T}$); $m(T) = m(\breve{T})$ since throughout the construction of $\breve{T}$ from T, all links and weights are preserved.



A transformed proof-structure $\breve{T}$ is again a cluster by the following (i) and (ii):
(i) $\breve{T}$ is sequentializable: let us fix an arbitrary sequentialization of T starting with a removal of the principal link of conclusion $A\&B$. Every splitting whose preceding splittings do not contain a removal of $L(E_k)$ has a corresponding splitting in $\breve{T}$ by virtue of the above (†), and so does every removal of $L(E_k)$ itself. And after a removal of $L(E_k)$ in T and in $\breve{T}$, by successive splittings of $⅋$-links including M,

the same proof-structure is reached from T and $\breve{T}$. (ii) Every sequentialization of $\breve{T}$ starts with a removal of a &-link: For a new conclusion E_m in $\breve{T}$, a link whose conclusion is E_m in $\breve{T}$ is not a &-link in this case (b), hence cannot be removed in a cluster U_{i_m}, hence in $\breve{T}$. Other conclusions C of $\breve{T}$ have the corresponding conclusions of T and moreover every possible removal of a link with conclusion C has a corresponding removal in T.

It is automatic to check that the assertion of the proposition for T is equivalent to that for $\breve{T}$ since all principal links of $\breve{T}$ coincide with those of T (since E_m is never a principal &-link in this case (b)). On the other hand, $\breve{T}$ in turn has a non-terminal $\otimes$-link of weight 1 coming from a terminal $\otimes$-link $L(E_{m-1})$ of T. Hence by using Preliminary Reduction 2, $\breve{T}$ is reduced to $\breve{T}'$ such that the assertion for $\breve{T}'$ implies that for $\breve{T}$. The reduction yields that $n_+(\breve{T}') < n_+(\breve{T})$ together with $n(\breve{T}') = n(\breve{T})$, thus $m(\breve{T}') < m(\breve{T})$. Since $m(\breve{T}) = m(T)$, I.H. implies that the assertion holds for $\breve{T}'$. This implies it also holds for $\breve{T}$ (cf. Preliminary Reduction 2), hence for T (cf. Preliminary Reduction 3). This contradicts that $T \in H$.

End of Proof for Claim

To sum up for (Step m), in either cases (Case 1)$_m$ and (Case 2)$_m$, we have a path $C_m \sim_{S_{i_m}} E_m \sim L(E_m)$ for a conclusion E_m of a cluster U_{i_m} and a terminal $\otimes$-link $L(E_m)$ in the end-part hereditarily below E_m so that $L(E_m)$ is different from any $L(E_k)$ with $k \in \{1, \ldots, m-1\}$. By virtue of Remark 8, there must exist hereditarily above $L(E_m)$ a cluster $U_{i_{m+1}}$ such that $i_{m+1} \notin \{i, i_1, \ldots, i_m\}$, which yields the next step $m+1$. See the following Figure 3.

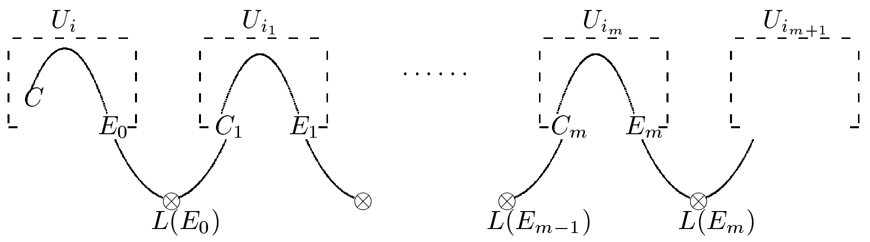


Figure 3: After Step m

Finally, we claim that (Step n) gives rise to a contradiction since the case yields a set $\{i, i_1, i_2, \ldots, i_n\}$ of indices of cardinality $n+1$, which contradicts with the number n of components $U_1, \ldots, U_n$. $\square$

Our next concern is to derive an important corollary (Corollary 4) of Proposition 3, which will be crucially used in proving our splitting lemma of the next section. For the corollary we prepare a lemma (Lemma 6) on a p-s's counterpart of a well known MALL provable equivalence $(X \& Y) \parr Z \equiv (X \parr Z) \& (Y \parr Z)$, which indicates a distribution of & over $\parr$.

First, we define a proof-structure $\widetilde{\Theta}$: Given a sequentializable p-s Θ of conclusions $X \& Y, Z, \Delta$ such that $X \& Y$ is a conclusion of a terminal (hence, principal) &-link L, we shall transform Θ into a sequentializable p-s $\widetilde{\Theta}$ of conclusions $(X \parr Z) \& (Y \parr Z), \Delta$ such that $(X \parr Z) \& (Y \parr Z)$ is a conclusion of a terminal &-link $\widetilde{L}$: First from Θ we obtain sequentializable p-s's $\Theta_1 := \Theta[p_L = 1]$ and $\Theta_0 := \Theta[p_L = 0]$ respectively of conclusions X, Z, Δ and Y, Z, Δ. Note that at this point the boundary $\partial(\Theta_1, \Theta_0)$ (cf. Definition 9) is determined to yield Θ from Θ_1 and Θ_0 by means of &-inference. In each sequentializable p-s Θ_1 and Θ_0 by drawing a terminal $⅋$-link, we obtain sequentializable p-s's, say $\bar{\Theta}_1$ and $\bar{\Theta}_0$, respectively of conclusions $X ⅋ Z, \Delta$ and $Y ⅋ Z, \Delta$. Our sequentializable p-s $\widetilde{\Theta}$ is defined to be the one obtained from $\bar{\Theta}_1$ and $\bar{\Theta}_0$ by means of &-rule by taking a boundary $\partial(\bar{\Theta}_1, \bar{\Theta}_0)$ to be obtained from $\partial(\Theta_1, \Theta_0)$ by deleting all the occurrences hereditarily above (and including) Z.

Second, we define a proof-structure $\widehat{\Theta}$ by using the above construction, which will allow an application of the distribution in a sequentializable p-s with respect to a &-link L whose weight is 1 and whose conclusion in turn is a premise of a $⅋$-link: For a given p-s Θ with a &-link L whose weight is 1 and whose conclusion $X\&Y$ is a premise of a $⅋$-link L', hence of conclusion $(X\&Y) ⅋ Z$ or $Z ⅋ (X\&Y)$, we shall define a sequentializable p-s $\widehat{\Theta}$ by the following procedures (i) $\sim$ (iii): (i) By replacing every $\otimes$-link hereditarily below L with a $⅋$-link, a p-s Θ_0 is obtained. (ii) Since $w(L) = 1$, all links hereditarily below L' (including L') in Θ_0 are $\{⅋, \oplus_1, \oplus_2\}$-links of weighs 1, hence are successively removed, which yields a p-s Θ_0^*, which has a $X\&Y$ and Z as two different conclusions. (iii) Then with respect to two conclusions $X\&Y$ and Z of Θ_0^*, the above construction is applied to obtain a p-s $\widetilde{\Theta_0^*}$. Then our proof-structure $\widehat{\Theta}$ is obtained from $\widetilde{\Theta_0^*}$ first by recovering removed $\{⅋, \oplus_1, \oplus_2\}$-links except L' in (ii) and second by changing $⅋$-links replaced in (i) with $\otimes$-links.

For such sequentializable p-s's $\widetilde{\Theta}$ and $\widehat{\Theta}$ obtained from Θ, we have the following lemma, which is easily derived by careful analysis of the algorithm of the adequacy theorem (Lemma 2). We say that *a p-s Θ yields $A \sim B$* if for some switching S, $A \sim_S B$ holds in the graph Θ_S.

Lemma 6 (distribution $(X\&Y) ⅋ Z \equiv (X ⅋ Z)\&(Y ⅋ Z)$ in sequentializable p-s's)

(i) If Θ is a cluster with a principal link L, then so is $\widetilde{\Theta}$ with a principal link $\widetilde{L}$. If Θ is a cluster, then so is $\widehat{\Theta}$ and all the principal links of Θ coincide with those of $\widehat{\Theta}$.

(ii) The followings hold for $\widetilde{\Theta}$:

(a) For A and B in Δ, if $\widetilde{\Theta}$ yields $A \sim B$ then so does Θ.

(b) For A in Δ if $\widetilde{\Theta}$ yields $(X ⅋ Z)\&(Y ⅋ Z) \sim A$ then Θ yields either $X\&Y \sim A$ or $Z \sim A$.

(iii) The following holds for $\widehat{\Theta}$:

- *If the assertion of Proposition 3 holds for a cluster $\widehat{\Theta}$, then so does for a cluster Θ.*

Proof (i) Straightforward.

For the proof of (ii)-(a) and (ii)-(b), first note that given a switching S for $\widetilde{\Theta}$ such that $S(\widetilde{L})$ is normal, there exists a corresponding switching $\underline{S}$ for Θ such that the graphs $\Theta_{\underline{S}}$ and $\widetilde{\Theta}_S$ are the same except that the latter has one additional node $X ⅋ Z$ together with an edge connecting to either X or to Y.

In the following for a given switching S for $\widetilde{\Theta}$, S_0 denotes S except that $S_0(\widetilde{L})$ is normal.

(ii)-(a) Suppose $A \sim_S B$. Since $(X ⅋ Z)\&(Y ⅋ Z)$ is a different conclusion from A and B of $\widetilde{\Theta}$, we have $A \sim_{S_0} B$ in $\widetilde{\Theta}$. Now the assertion follows from the first note.

(ii)-(b) Suppose $A \sim_S (X ⅋ Z)\&(Y ⅋ Z)$.

(Case 1) The case where $S(\widetilde{L})$ is normal:

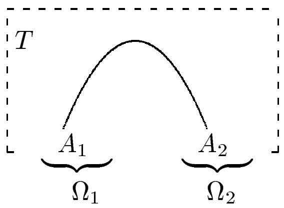
In this case the supposition implies either $A \sim_S X ⅋ Z$ or $A \sim_S Y ⅋ Z$. Hence

from the first note we have in Θ either $A \sim_{\underline{S}} X$, $A \sim_{\underline{S}} Z$ or $A \sim_{\underline{S}} Y$, $A \sim_{\underline{S}} Z$, thus we conclude that either $X \& Y \sim_{\underline{S}} A$ or $Z \sim_{\underline{S}} A$ holds.
(Case 2) The case where $S(\widetilde{L})$ is proper:
Let K denote the link of the conclusion $S(\widetilde{L})$ in $\varphi_S(\Theta)$. Then K depends on $p_{\widetilde{L}}$. From the supposition we have $A \sim_{S_0} S(\widetilde{L})$.
(Case 2.1) The case where $(X \parr Z)\&(Y \parr Z)$ is a hereditary conclusion of K:
In this case $S(\widetilde{L})$ is a sub-formula of $(X \parr Z)\&(Y \parr Z)$. Hence we change S_0 into S_0' by appropriate choice of $\parr$-switchings between $S(\widetilde{L})$ and $(X \parr Z)\&(Y \parr Z)$ so that $S(\widetilde{L}) \sim_{S_0'} (X \parr Z)\&(Y \parr Z)$ holds. It is important that the change does not affect the path between $A \sim_{S_0} S(\widetilde{L})$, thus we still have $A \sim_{S_0'} S(\widetilde{L})$. Hence we have $A \sim_{S_0'} (X \parr Z)\&(Y \parr Z)$. Since $S_0'(\widetilde{L})$ is normal, the assertion follows from Case 1.
(Case 2.2) Negation of (Case 2.1):
In this case by Remark 2 (ii), K must be hereditarily above $\partial(\bar{\Theta}_1, \bar{\Theta}_0)$, hence by Remark 2 (i) K has a unique inverse image, say K^*, in one of $\bar{\Theta}_1$ and $\bar{\Theta}_0$, thus in this case the link K^* occurs in one of Θ_1 and Θ_0. The above definition of $\partial(\bar{\Theta}_1, \bar{\Theta}_0)$ guarantees that the image of K^* in Θ is hereditarily above $\partial(\Theta_1, \Theta_0)$, hence depends on an eigenweight p_L again by Remark 2 (ii). On the other hand since $A \sim_{S_0} S(\widetilde{L})$, by the first note in Θ we have $A \sim_{\underline{S_0}} S(\widetilde{L})^*$, where $S(\widetilde{L})^*$ is a conclusion of K^* in Θ. Now we change the switching $\underline{S_0}$ into a new one $(\underline{S_0})'$ such that $(\underline{S_0})'$ is $\underline{S_0}$ except that a jump $(\underline{S_0})'(L)$ is $S(\widetilde{L})^*$; this choice of jump is possible since K^* depends on p_L. Since the switching $(\underline{S_0})'$ for Θ yields an edge between $X\&Y$ and $S(\widetilde{L})^*$ together with $A \sim_{(\underline{S_0})'} S(\widetilde{L})^*$, we have $A \sim_{(\underline{S_0})'} X\&Y$ in Θ.
(iii) Straightforward from *(ii)* by virtue of the construction of $\widehat{\Theta}$.

$\square$

With the help of Lemma 6, we derive the following corollary of Proposition 3, which will be crucial in proving our splitting lemma in the next section.

Corollary 4 (Corollary of main proposition (Proposition 3)) *For a &-cluster T, let Ω_1 and Ω_2 denote disjoint and non-empty lists of conclusions of T such that any conclusion of terminal &-links in T is either in Ω_1 or in Ω_2. Then there exist A_1 in Ω_1 and A_2 in Ω_2 such that T yields $A_1 \sim A_2$.*



Proof By induction on the number of formulas which Ω_1 and Ω_2 contain.
(Base Case:) The case where each of Ω_1 and Ω_2 consists only of one formula.
The assertion is direct from Proposition 3.
(Induction Case:) If the following (1) or (2) holds, the assertion is direct from Proposition 3: (1) Either Ω_1 or Ω_2 consists only of one formula. (2) Either Ω_1 or Ω_2 does not contain any conclusion of terminal &-links.
Hence we may assume that Ω_1 contains different occurrences $X\&Y$ and Z such that $X\&Y$ is a conclusion of a terminal &-link, say L. From T we obtain $\widetilde{T}$ with conclusions $(X \parr Z)\&(Y \parr Z), \Delta$. From Lemma 6 *(i)*, $\widetilde{T}$ is still a &-cluster.

Let $\widetilde{\Omega}_1$ denote the list obtained from Ω_1 by changing all occurrences $X \& Y$, Z into occurrences $(X \parr Z) \& (Y \parr Z)$, which is a conclusion of a terminal &-link $\widetilde{L}$ in $\widetilde{T}$. The induction hypothesis can be applied to the cluster $\widetilde{T}$ with lists $\widetilde{\Omega}_1$ and Ω_2, whose number of formulas is less than that of Ω_1 and Ω_2. Thus from I.H. there exist $\widetilde{A}_1$ in $\widetilde{\Omega}_1$ and A_2 in Ω_2 such that $\widetilde{T}$ yields $\widetilde{A}_1 \sim A_2$.

- If $\widetilde{A}_1$ is not $(X \& Y) \parr Z$, then $\widetilde{A}_1$ is in Ω_1. Then the assertion follows from Lemma 6 (ii)-(a).

- If $\widetilde{A}_1$ is $(X \& Y) \parr Z$, then from Lemma 6 (ii)-(b), T yields either $X \& Y \sim A_2$ or $Z \sim A_2$, hence the assertion has been derived since $X \& Y$ and Z are in Ω_1. $\square$

4 MALL+Mix Sequentialization Theorem

In this section we shall prove a sequentialization theorem for MALL+Mix (Theorem 3), which states that a MALL proof-structure Θ is sequentializable for MALL+Mix if and only if for all switchings S, the graph Θ_S is acyclic. The criterion is obtained from Girard's [9] for MALL just by dropping connectedness, hence is the canonical extension of the Danos/Fleury-Retoré MLL+Mix-criterion [6]. Although it is very natural to expect the criterion in the presence of Girard's MALL-criterion and the Danos/Fleury-Retoré MLL+Mix-criterion, this result has not been previously demonstrated. We should point out that in an entirely different framework, using additive-boxes [7], Tortora mentions that the proof of Danos for MLL+Mix could be generalized into MALL+Mix (cf.Theorem 2.9 of [15]). Our proof of the MALL+Mix sequentialization is carried out by using to significant advantage our characterization of proof-structures by means of softness, which we have investigated in Section 2. For the sequentialization theorem, the main technical lemma is the splitting lemma (Proposition 4) for *simple proof-structures* (Definition 16). The splitting lemma is proved by using geometric properties of clusters (Corollary 4 of Proposition 3), which we have investigated in Section 3.

We start with defining a notation $\Theta\lceil L_1, \ldots, L_n \rceil$ for a proof-structure:

Definition 15 (The proof-structure $\Theta\lceil L_1, \ldots, L_n \rceil$) Given a proof-structure Θ and a set $\{L_1, \ldots, L_n\}$ of $\parr$-links in it, $\Theta\lceil L_1, \ldots, L_n \rceil$ denotes the proof structure obtained from Θ by replacing $\parr$-links $L_1, \ldots, L_n$ (together with inherited occurrences of $\parr$ in descendant formulas) with $\otimes$-links (together with occurrences of $\otimes$). *The image of* L_i denotes the $\otimes$-link in $\Theta\lceil L_1, \ldots, L_n \rceil$ obtained from L_i via the replacement.

Given an arbitrary proof-structure Θ, if $L_1, \ldots, L_n$ are $\parr$-links in $|\Theta|$ (cf. Definition 5) whose inverse image in Θ under the map $|\ \ |$ are $\otimes$-links, then evidently:

$$\Theta = |\Theta|\, \lceil L_1, \ldots, L_n \rceil \tag{3}$$

Note that in (3) all the $\otimes$-links in Θ are the images of $L_1, \ldots, L_n$. Moreover from Corollary 2, we know that $|\Theta|$ is PALL+Mix sequentializable.

4.1 Splitting Lemma

This subsection is devoted to proving the splitting lemma (Proposition 4) for *simple proof-structures*, which will yield our MALL+Mix sequentialization theorem (Theorem 3) in the next subsection:

Definition 16 (simple proof-structure $\Theta_\pi\lceil I^\bullet\rceil := \Upsilon_\pi$ with the distinguished $\otimes$-link $L^@$) Let π denote a MALL+Mix proof π whose last inference I is a $\parr$-inference;

$$\pi \left\{ \begin{array}{c} \vdots \\ \dfrac{\vdash \Sigma, A, B}{\vdash \Sigma, A \parr B}\, I \end{array} \right.$$

Let $I^\bullet$ denote the $\parr$-link interpreting I in Θ_π, then $I^\bullet$ is a terminal $\parr$-link in Θ_π. A *simple* proof-structure is a proof-structure $\Theta_\pi\lceil I^\bullet\rceil$ (cf. Definition 15). $\Theta_\pi\lceil I^\bullet\rceil$ is simply denoted by Υ_π. In Υ_π we denote by $L^@$ the terminal $\otimes$-link which is the image of $I^\bullet$. Since by Remark 3 any terminal $\parr$-link can be removed and removal of a terminal link is closed in the class of sequentializable p-s's, we alternatively define as follows: A *simple* proof-structure is a proof-structure $\Theta\lceil L\rceil$ such that L is a terminal $\parr$-link of a MALL+Mix sequentializable proof-structure Θ.

Example 4 The left proof-structure below is simple since it is $\Theta\lceil L\rceil$ with a MALL+Mix sequentializable Θ (together with L) in the right-hand side:

$$A^\perp \quad \dfrac{A \quad B}{A \otimes B} \quad B^\perp \qquad\qquad \Theta := A^\perp \quad \dfrac{A \quad B}{A \parr B}\, L \quad B^\perp$$

Now the concept of clusters-representation of Definition 14 can be naturally extended from Θ_π to a simple proof-structure Υ_π:

Definition 17 (clusters-representation for a simple proof-structure) Consider a clusters-representation of Θ_π. Since $I^\bullet$ is terminal in Θ_π, it lies in the end-part (but not in any components) for the representation. Thus the replacement of $I^\bullet$ in Θ_π does not affect any components, whose occurrences are retained in $\Theta_\pi\lceil I^\bullet\rceil$ as sub-proof-structures. In $\Theta_\pi\lceil I^\bullet\rceil$ those occurrences which are not contained in any component are obtained from the end-part of Θ_π by the replacement of $I^\bullet$ with the $\otimes$-link $L^@$. To sum up, $\Theta_\pi\lceil I^\bullet\rceil$ is represented from a union of clusters by drawing $\{\otimes, \parr, \oplus_1, \oplus_2\}$-links of weights 1 below them, where the drawing of the $\otimes$-link $L^@$ does not necessarily interpret any $\otimes$-inference as opposed to the other drawings which do. We say this representation as *clusters-representation* for a simple proof-structure. *Components* and *end-part* of a representation are defined in the same manner as a representation of a sequentializable p-s (cf. Definition 14).

Our splitting lemma then states:

Proposition 4 (splitting lemma for a simple proof-structure Υ_π) *For a simple proof-structure $\Theta := \Upsilon_\pi$, consider the following three conditions:*

(i) All terminal links in Θ are $\otimes$-links.
(ii) For all switchings S, Θ_S is acyclic.
(iii) In Θ, there is a terminal $\otimes$-link which can be removed.

Then $(i) \wedge (ii) \Rightarrow (iii)$.

Proof Throughout this proof a proof-structure always means a proof-structure with a clusters-representation. We prove the assertion by induction on the number of components in a simple proof-structure $\Theta := \Upsilon_\pi$, where π is

$$\begin{array}{c} \vdots \\ \dfrac{\vdash \Sigma, A, B}{\vdash \Sigma, A \parr B}\, I \end{array}.$$

(**Base Case**)[2]: The case where Θ consists of only one component T_1 which is a cluster. In this case, we shall prove that $(i) \Rightarrow \neg(ii)$, which implies the assertion.
We prove the assertion when T_1 is a &-cluster since the assertion is obvious for an axiom-cluster. In this case, among possible proofs π such that $\Theta = \Upsilon_\pi$, we can choose π whose lower-most binary inference, say R, is & (hence the link interpreting R is principal in T_1) and all inferences below R and above I are $\{⅋, \oplus\}$.

Let π' denote the subproof whose last inference is R, then $T_1 = T_{\pi'}$:

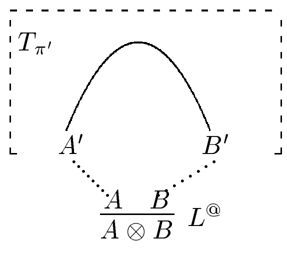
$$\pi' \left\{ \begin{array}{c} \begin{array}{cc} \vdots & \vdots \\ \vdash \Delta, E & \vdash \Delta, F \end{array} \\ \hline \vdash \Delta, E\&F \end{array} \right. R$$

$$\begin{array}{c} \vdots \\ \vdash \Sigma, A, B \\ \hline \vdash \Sigma, A ⅋ B \end{array} I$$

Let Ω_1 (respectively Ω_2) denote T_1's conclusions which are *ancestors* of A (respectively of B). Then Ω_1 and Ω_2 are non-empty and disjoint. Moreover from the condition (i) for Υ_π, every conclusion (particularly $E\&F$) of terminal &-links in T_1 is either in Ω_1 or in Ω_2; otherwise in Υ_π there arises a terminal $\{\&, ⅋, \oplus_1, \oplus_2\}$-link which is hereditarily below (or is itself) a terminal &-link in T_1. Thus Corollary 4 in the previous section implies that there exists A' in Ω_1 and B' in Ω_2 such that $A' \sim_S B'$ for some switching S for T_1. Since A' and B' are subformulas respectively of A and of B and the $\otimes$-link $L^@$ is in the end-part, there exist subformula edges determined by the choices of ⅋-switchings between A and A' and between B and B'. Thus we have a cycle in the graph Θ_S, whose path is specified as follows:

$$A \otimes B, A, \ldots, A' \sim_S B', \ldots, B, A \otimes B$$

See the following figure for the cycle in a proof-structure Υ_π. Thus we have derived $\neg(ii)$ in this case.



(**Induction Case**): In this case we shall prove $(i) \wedge \neg(iii) \Rightarrow \neg(ii)$, which is equivalent to the assertion. We suppose (i) and $\neg(iii)$ for deriving $\neg(ii)$.
In this case Θ_π consists of at least two components. Hence among possible MALL+Mix proofs π such that $\Theta = \Upsilon_\pi$, we can choose such π whose lower-most binary inference, say R, is a $\otimes$ or a Mix rule. Note that if R is $\otimes$-rule, then the link $R^\bullet$ interpreting R is in the end-part of a representation Θ_π, hence is in that of Υ_π. Moreover in this case the principal formulas of all inferences below R, which are $\{⅋, \oplus\}$-rules, must be ancestors of $A ⅋ B$; otherwise in Υ_π there arises a terminal $\{⅋, \oplus_1, \oplus_2\}$-link, which contradicts (i). Thus we describe the conclusion of R as follows where A_i

[2] This is the case where Corollary 4 of Proposition 3 is used.

(respectively B_i) is an ancestor of A (respectively of B):

$$\dfrac{\dfrac{\vdots\,\pi_1 \qquad\qquad \vdots\,\pi_2}{\vdash \Sigma, A_1, \ldots, A_n, B_1, \ldots, B_m}\,R}{\vdots} \qquad \dfrac{\vdash \Sigma, A, B}{\vdash \Sigma, A \parr B}\,I$$

We often denote by $< \pi_1, \pi_2 >$ the proof obtained from π_1 and π_2 by applying the inference R.
Our proof is divided into the following three cases. The crucial idea is to consider in each case simple proof-structures to be denoted by Θ_k and by Θ^k $(k = 1, 2)$ whose components are those of Θ such that

$$\text{number of components in } \Theta = \sum_{k=1,2} \text{number of components in } \Theta_k = \sum_{k=1,2} \text{number of components in } \Theta^k$$

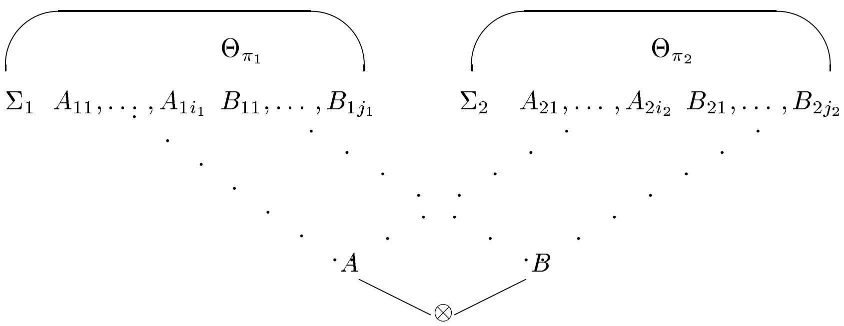
(**Case 1**) R is Mix.

$$\dfrac{\begin{array}{c}\vdots\,\pi_1 \\ \vdash \Sigma_1, A_{11}, \ldots, A_{1i_1}, B_{11}, \ldots, B_{1j_1}\end{array} \quad \begin{array}{c}\vdots\,\pi_2 \\ \vdash \Sigma_2, A_{21}, \ldots, A_{2i_2}, B_{21}, \ldots, B_{2j_2}\end{array}}{\vdash \Sigma_1, \Sigma_2, A_{11}, \ldots, A_{1i_1}, B_{11}, \ldots, B_{1j_1}, A_{21}, \ldots, A_{2i_2}, B_{21}, \ldots, B_{2j_2}}\,(Mix)$$

$$\vdots$$

$$\dfrac{\vdash \Sigma_1, \Sigma_2, A, B}{\vdash \Sigma_1, \Sigma_2, A \parr B}\,I$$

Then the proof-structure Θ is of the following form:



The proof-structure Θ in Case 1

The proof for this case is a special case of that for the following Case 2. Thus we go into Case 2:

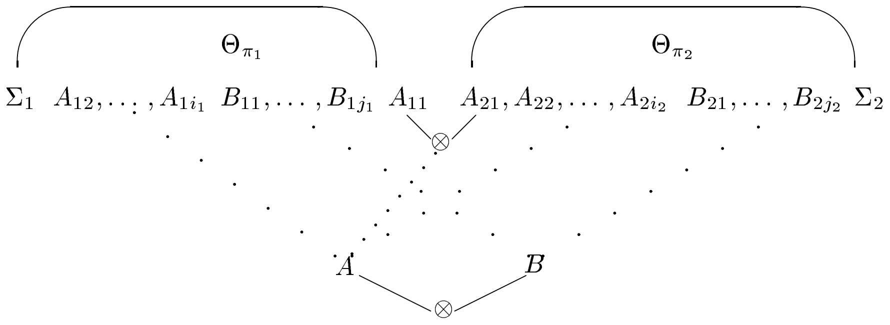
(**Case 2**) R is a $\otimes$-inference whose principal formula is an ancestor of $A \parr B$; i.e., is A_i $(i = 1, \ldots, n)$ or B_j $(j = 1, \ldots, m)$. Without loss of generality we may assume that it is A_1:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1 \\ \vdash \Sigma_1, A_{11}, A_{12}, \ldots, A_{1i_1}, B_{11}, \ldots, B_{1j_1}\end{array} \quad \begin{array}{c}\vdots\,\pi_2 \\ \vdash \Sigma_2, A_{21}, A_{22}, \ldots, A_{2i_2}, B_{21}, \ldots, B_{2j_2}\end{array}}{\vdash \Sigma_1, \Sigma_2, A_{11} \otimes A_{21}, A_{12}, \ldots, A_{1i_1}, B_{11}, \ldots, B_{1j_1}, A_{22}, \ldots, A_{2i_2}, B_{21}, \ldots, B_{2j_2}}\,(\otimes)$$

$$\vdots$$

$$\dfrac{\vdash \Sigma_1, \Sigma_2, A, B}{\vdash \Sigma_1, \Sigma_2, A \parr B}\,I$$

Then the proof-structure Θ is of the following form:

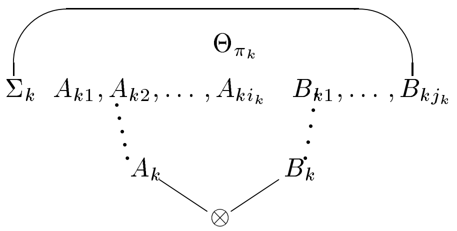


The proof-structure Θ in Case 2

In this case consider the following proof π'_k $(k = 1, 2)$ which is obtained from π_k by skipping R and the inferences (below R) such that at least one *auxiliary* formula does not contain any descendent of $A_{k1}, \ldots, A_{ki_k}, B_{k1}, \ldots, B_{kj_k}$: (If $i_k = 0$ or $j_k = 0$ (i.e., $i_k \cdot j_k = 0$). We define π'_k in the same way, except without the final inference I by noting that A_k or B_k does not exist.)

$$
\begin{array}{c}
\vdots\ \pi_k \\
\vdash \Sigma_k, A_{k1}, A_{k2}, \ldots, A_{ki_k}, B_{k1}, \ldots, B_{kj_k} \\
\vdots \\
\dfrac{\vdash \Sigma_k, A_k, B_k}{\vdash \Sigma_k, A_k \parr B_k}\ I
\end{array}
$$

If $i_k \cdot j_k \neq 0$ (respectively, $i_k \cdot j_k = 0$), let Θ_k denote the simple proof-structure $\Upsilon_{\pi'_k}$ (respectively, the sequentializable p-s $\Theta_{\pi'_k}$) with the clusters-representation where each component is that of Θ:



The proof-structure Θ_k in Case 2 (with $i_k \cdot j_k \neq 0$)

We have (number of components in Θ_k) $<$ (number of components in Θ) since R is a binary inference.
For convenience, when $i_k \cdot j_k = 0$, we declare that the $\otimes$-link $L^{@}$ in Θ_k can be removed despite the absence of the link $L^{@}$ in Θ_k. The definition is consistent with the argument to follow.

The following is a crucial claim in this case:
(Claim:) If $\forall k \in \{1, 2\}$ the $\otimes$-link $L^{@}$ in Θ_k can be removed, then the $\otimes$-link $L^{@}$ in Θ can be removed.

Proof of Claim: If $L^{@}$ in Θ_k can be removed then Θ_{π_k} is a union of two proof-structures, one, say ${}_1\Theta_k$, has conclusions ${}_1\Sigma_k, A_{k1}, \ldots, A_{ki_k}$ and the other, say ${}_2\Theta_k$, has conclusions ${}_2\Sigma_k, B_{k1}, \ldots, B_{ki_k}$ with $\Sigma_k = {}_1\Sigma_k,\ {}_2\Sigma_k$. Then $\Theta_{<\pi_1, \pi_2>}$ is the union of two proof-structures: one is obtained from the union of ${}_1\Theta_1$ and ${}_1\Theta_2$ by drawing a $\otimes$-link of premises A_{11} and A_{21} (hence with conclusions ${}_1\Sigma_1$,

${}_1\Sigma_2, A_{11} \otimes A_{21}, A_{12}, \ldots, A_{1i_1}, A_{22}, \ldots, A_{2i_2})$, and the other is the union of ${}_2\Theta_1$ and ${}_2\Theta_2$ (hence with conclusions ${}_2\Sigma_1$, ${}_2\Sigma_2, B_{11}, \ldots, B_{1j_1}, B_{21}, \ldots, B_{2j_2}$). This implies that $L^{@}$ in Θ can be removed. <u>End of Proof for Claim</u>

With the supposition $\neg(iii)$ for Θ, the above claim implies that $\exists\, k_0 \in \{1, 2\}$ in Θ_{k_0} such that $L^{@}$ cannot be removed. Note that for every terminal $\otimes$-link other than $L^{@}$ of Θ_k, it can be removed iff the corresponding terminal $\otimes$-link in Θ can be removed. Hence the supposition $\neg(iii)$ for Θ implies $\neg(iii)$ for Θ_{k_0}. Since the condition (i) for Θ_{k_0} is automatically satisfied by construction, the induction hypothesis implies that $\neg(ii)$ holds for Θ_{k_0}. Now we show that $\neg(ii)$ for Θ_{k_0} implies $\neg(ii)$ for Θ as shown in Figure 4: Suppose $\neg(ii)$ holds for Θ_{k_0}; i.e., there is a cycle in a graph resulting from Θ_{k_0}. We may assume that the cycle in Θ_{k_0} passes through the premises (A_{k_0} and B_{k_0}) and the conclusion of $L^{@}$; otherwise a cycle obviously appears in Θ. Then $\exists i \in \{1, \ldots, i_{k_0}\}$ $\exists j \in \{1, \ldots, j_{k_0}\}$ there is a path, say p, on the cycle between $A_{k_0 i}$ and $B_{k_0 j}$ under a certain switching for $\Theta_{\pi_{k_0}}$, which is a sub-proof-structure of Θ. The switching can be extended to a switching for Θ retaining the path p in such a way that there is a path coming from the subformula relation between A (respectively B) and $A_{k_0 i}$ (respectively $B_{k_0 j}$) determined by $\parr$-switchings. Now a cycle can be drawn as follows in the graph resulting from Θ. Hence we have derived $\neg(ii)$ for Θ.

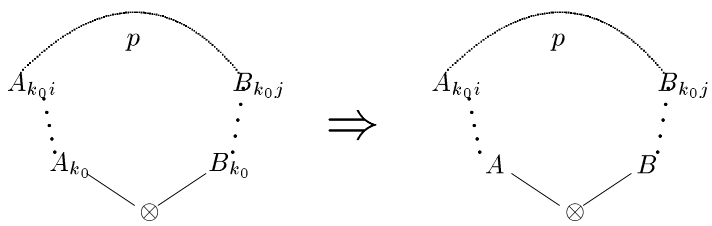


Figure 4: A cycle in Θ_{k_0} implies a cycle in Θ.

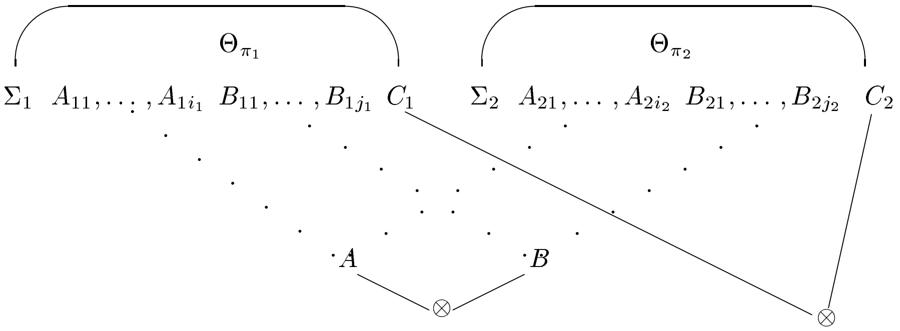
The following Case 3 is the most crucial case in the proof:

(**Case 3**) R is a $\otimes$-inference whose principal formula is not an ancestor of $A \parr B$; i.e., is in Σ.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash \Sigma_1, C_1, A_{11}, \ldots, A_{1i_1}, B_{11}, \ldots, B_{1j_1}\end{array} \quad \begin{array}{c}\vdots\ \pi_2\\ \vdash \Sigma_2, C_2, A_{21}, \ldots, A_{2i_2}, B_{21}, \ldots, B_{2j_2}\end{array}}{\vdash \Sigma_1, \Sigma_2, C_1 \otimes C_2, A_{11}, \ldots, A_{1i_1}, B_{11}, \ldots, B_{1j_1}, A_{21}, \ldots, A_{2i_2}, B_{21}, \ldots, B_{2j_2}}\ (\otimes)$$

$$\begin{array}{c}\vdots\\ \dfrac{\vdash \Sigma_1, \Sigma_2, C_1 \otimes C_2, A, B}{\vdash \Sigma_1, \Sigma_2, C_1 \otimes C_2, A \parr B}\ I\end{array}$$

Then the proof-structure Θ is of the following form:



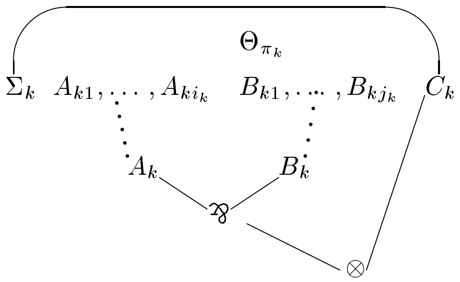
The proof-structure Θ in Case 3

Our proof in this case consists of the following two steps. In (Step 3.k) with $k = 1$ and $k = 2$, we shall consider a proof-structure to be denoted by Θ_k and Θ^k, respectively.

(**Step 3.1**) Consider the following proof π'_k $(k = 1, 2)$ which is obtained from π by skipping R and the inferences (below R) such that at least one *auxiliary* formula does not contain any descendent of $A_{k1}, \dots , A_{ki_k}, B_{k1}, \dots , B_{kj_k}$. (If $i_k = 0$ or $j_k = 0$ the second to last $⅋$-inference does not exist since either A_k or B_k does not exist.):

$$
\dfrac{\dfrac{\begin{array}{c} \vdots\ \pi_k \\ \vdash \Sigma_k, C_k, A_{k1}, \dots , A_{ki_k}, B_{k1}, \dots , B_{kj_k} \\ \vdots \\ \vdash \Sigma_k, C_k, A_k, B_k \end{array}}{\vdash \Sigma_k, C_k, A_k ⅋ B_k}}{\vdash \Sigma_k, C_k ⅋ (A_k ⅋ B_k)}\ I
$$

Let Θ_k denote the simple proof-structure $\Upsilon_{\pi'_k}$ with the clusters-representation in which each component is that in Θ:



The proof-structure Θ_k in Step 3.1 (with $i_k \cdot j_k = 0$)

We have (number of components in Θ_k) $<$ (number of components in Θ).
The following is a crucial claim in this step:

(Claim 1:) If $\exists k \in \{1, 2\}$ such that in Θ_k the $\otimes$-link $L^{@}$ (of conclusion $(A_k ⅋ B_k) \otimes C_k$) can be removed, then in Θ the terminal $\otimes$-link of the conclusion $C_1 \otimes C_2$ can be removed.

<u>Proof for Claim 1</u>: If $L^{@}$ in Θ_k can be removed, then Θ_{π_k} is a union of two proof-structures, one of which, say Θ', has conclusions C_k, Σ'_k with $\Sigma'_k \subseteq \Sigma_k$. This implies that the terminal $\otimes$-link of the conclusion $C_1 \otimes C_2$ can be removed to obtain two proof-structures, one of which is Θ'. <u>End of Proof for Claim 1</u>

Note that a terminal $\otimes$-link other than $L^{@}$ of Θ_k can be removed iff the corresponding $\otimes$-link of Θ can be removed. Hence the condition $\neg(iii)$ for Θ together with Claim 1 implies $\neg(iii)$ for Θ_k for all $k \in \{1, 2\}$. Because the conditions (i) is automatically satisfied by the construction of Θ_k, the induction hypothesis implies that $\neg(ii)$ holds for Θ_k for all $k \in \{1,2\}$.

By d_k we denote a cycle in the graph obtained from Θ_k with $k \in \{1,2\}$. Now we show

(Claim 2:) If d_1 appears under the left (respectively the right) switching for $A_1 \parr B_1$ in Θ_1 and d_2 appears under the right (respectively the left) switching for $A_2 \parr B_2$ in Θ_2, then we can draw a cycle in Θ, hence $\neg(ii)$ holds for Θ (cf. Figure 5).

<u>Proof for Claim 2</u>: We consider the case where d_1 appears under the left switching and d_2 appears under the right switching. We may assume that $\forall k \in \{1,2\}$ d_k passes through the premises ($A_k \parr B_k$ and C_k) and the conclusion of $L^{@}$ in Θ_k; otherwise a cycle obvious appears in Θ. Then $\exists i \in \{1,\ldots,i_1\}$ such that there is a path p_1 on the cycle d_1 between A_{1i} and C_1 under a certain switching for Θ_{π_1} and $\exists j \in \{1,\ldots,j_2\}$ such that there is a path p_2 on the cycle d_2 between B_{2i} and C_2 under a certain switching for Θ_{π_2}. Given that Θ_{π_1} and Θ_{π_2} are disjoint sub-proof-structures of Θ, both switchings can be simultaneously extended to a switching for Θ to retain the appearance of both p_1 and p_2 and to make the appearance of the subformula edges between A (respectively B) and A_{1i} (respectively B_{2j}) determined by $\parr$-switchings. Now a cycle can be drawn as follows in the graph obtained from Θ.

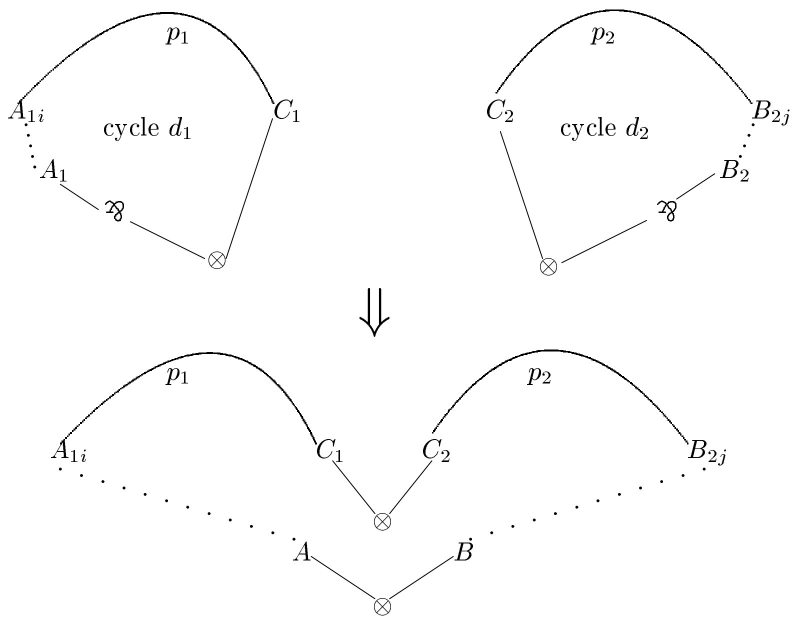


Figure 5: Cycles d_1 under the *left* switching for $A_1 \parr B_1$ and d_2 under the *right* switching for $A_2 \parr B_2$ imply a cycle in Θ.

<u>End of Proof for Claim 2</u>

By virtue of Claim 2, in what follows for our purpose to derive $\neg(ii)$ for Θ we may suppose the following (Supposition 1) without loss of generality;

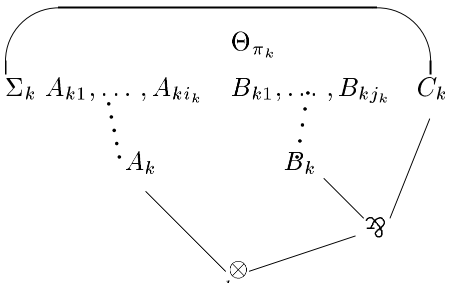
(Supposition 1:) The cycles d_1 and d_2 appear under both *right* switchings for $A_1 \parr B_1$ and for $A_2 \parr B_2$ in Θ_1 and in Θ_2, respectively.

(**Step 3.2**) Subject to Supposition 1, consider the following proof $\tilde{\pi}_k$ ($k = 1,2$) which is obtained from π by skipping R and the inferences (below R) such that at least one *auxiliary* formula does not contain any descendent of $A_{k1},\ldots A_{ki_k}, B_{k1},\ldots,B_{kj_k}$:

(If $i_k = 0$ we define $\tilde{\pi}_k$ in the same way, except without the final inference I.)

$$
\cfrac{\cfrac{\begin{array}{c} \vdots\, \pi_k \\ \vdash \Sigma_k, C_k, A_{k1}, \dots, A_{ki_k}, B_{k1}, \dots, B_{kj_k} \\ \vdots \end{array}}{\vdash \Sigma_k, A_k, B_k, C_k}}{\cfrac{\vdash \Sigma_k, A_k, B_k \parr C_k}{\vdash \Sigma_k, A_k \parr (B_k \parr C_k)}\; I}
$$

If $i_k \neq 0$ (respectively, $i_k = 0$), let Θ_k denote the simple proof-structure $\Upsilon_{\tilde{\pi}_k}$ (respectively, the sequentializable p-s $\Theta_{\tilde{\pi}_k}$) for whose clusters-representation each component is $\Theta := \Upsilon_\pi$:



The proof-structure Θ^k in Step 3.2 (with $i_k \neq 0$)

We have (number of components in Θ^k) $<$ (number of components in Θ).
For convenience, when $i_k = 0$, we declare that the $\otimes$-link $L^@$ in Θ_k can be removed despite the absence of the link $L^@$ in Θ_k. The definition is consistent with the argument to follow.

The following is a crucial claim in this step:

(Claim 3:) If $\forall\, k \in \{1,2\}$ in Θ^k the $\otimes$-link $L^@$ (of conclusion $A_k \otimes (B_k \parr C_k)$) can be removed, then the $\otimes$-link $L^@$ (of conclusion $A \otimes B$) in Θ can be removed.

<u>Proof for Claim 3</u>: If $L^@$ in Θ^k can be removed for all $k \in \{1,2\}$, then Θ_{π_k} is a union of two proof-structures, one, say ${}_1\Theta^k$, is of conclusions ${}_1\Sigma_k, A_{k1}, \dots, A_{ki_k}$ and the other, say ${}_2\Theta^k$, is of conclusions ${}_2\Sigma_k, B_{k1}, \dots, B_{kj_k}, C_k$ with $\Sigma_k = {}_1\Sigma_k$, ${}_2\Sigma_k$. Then $\Theta_{<\pi_1,\pi_2>}$ is a union of two proof-structures, one is a union of ${}_1\Theta^1$ and ${}_1\Theta^2$ (hence with conclusions, ${}_1\Sigma_1$, ${}_1\Sigma_2, A_{11} \dots, A_{1i_1}, A_{21}, \dots, A_{2i_2}$), and the other is obtained from a union of ${}_2\Theta^1$ and ${}_2\Theta^2$ by drawing a $\otimes$-link of premises C_1 and C_2 (hence with conclusions ${}_2\Sigma_1$, ${}_2\Sigma_2, B_{11} \dots, B_{1j_1}, B_{21}, \dots, B_{2j_2}, C_1 \otimes C_2$). This implies that $L^@$ in Θ can be removed. <u>End of Proof for Claim 3</u>

Since every $\otimes$-link other than $L^@$ of Θ^k can be removed iff the corresponding $\otimes$-link of Θ can be removed, the condition $\neg(iii)$ for Θ together with Claim 1 implies that $\exists k_0 \in \{1,2\}$ $\neg(iii)$ holds for Θ^{k_0}. Since the condition (i) is automatically satisfied by the construction of Θ^{k_0}, the induction hypothesis implies that $\neg(ii)$ holds for Θ^{k_0}. Thus we denote by h_{k_0} a cycle in the graph obtained from Θ^{k_0}. If h_{k_0} appears under the left switching for $B_{k_0} \parr C_{k_0}$, then we can draw a cycle in Θ to derive $\neg(ii)$ for Θ by a similar argument to Figure 4 of Case 2. See the following Figure 6: Hence in what follows without loss of generality we may suppose;

(Supposition 2:) The cycle h_{k_0} appears under the *right* switching for $B_{k_0} \parr C_{k_0}$ in Θ^{k_0}.

With Supposition 1 (of Step 3.1) and Supposition 2 we can draw a cycle in Θ from h_{k_0} and $d_{k'}$ with $\{k_0, k'\} = \{1,2\}$ as shown in Figure 7. On the one hand since $d_{k'}$ appears under the right switching for $A_{k'} \parr B_{k'}$ by Supposition 1, there is a path $q_{k'}$ between $B_{k'j}$ and $C_{k'}$ for some $j \in \{1, \dots, j_{k'}\}$. On the other hand since h_{k_0}

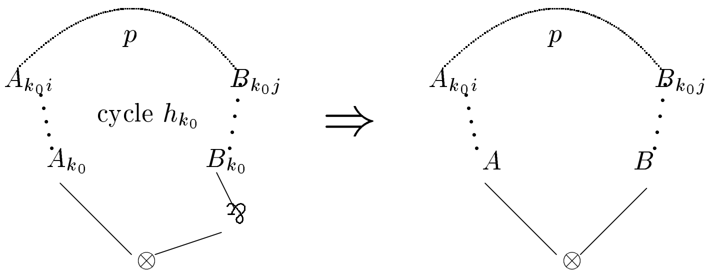


Figure 6: A cycle h_{k_0} under the left switching for $B_{k_0} \parr C_{k_0}$ implies a cycle in Θ.

appears under the right switching for $B_{k_0} \parr C_{k_0}$ by Supposition 2, there is a path q_{k_0} between $A_{k_0 i}$ and C_{k_0} for some $i \in \{1, \dots, i_{k_0}\}$. Then as shown in Figure 7, a cycle can be drawn in a graph obtained from Θ by the similar argument to Figure 5 of Step 3.1. Thus we have derived $\neg(ii)$ for Θ.

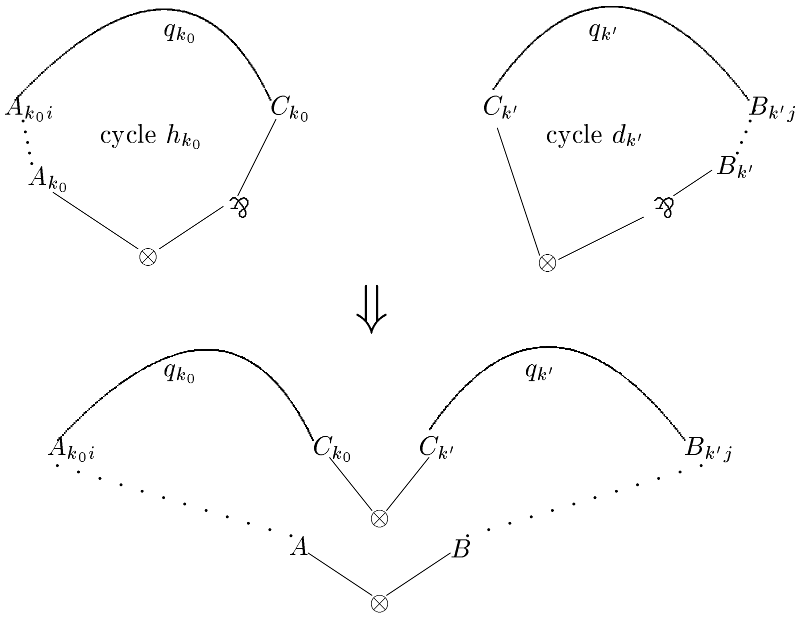


Figure 7: Cycles $d_{k'}$ of Supposition 1 and h_{k_0} of Supposition 2 imply a cycle in Θ

$\square$

If Θ of Proposition 4 is an MLL proof-structures, in which case components of a representation turn out to be all the axiom-links in Θ (cf. Definition 14), our proof of Proposition 4 can be seen as a proof for a multiplicative splitting lemma. This is why our proof of Theorem 3 below will contain an alternative proof of a MLL+Mix sequentialization theorem (cf. Remark 10 below).

4.2 MALL+Mix Sequentialization

Now we are ready to prove the main theorem in this section:

Definition 18 (MALL+Mix **proof-net)**
A MALL proof-structure Θ is called MALL+Mix *proof-net* if for every switching S the graph Θ_S is acyclic.

Remark 9 (removal of a terminal link is closed in MALL+Mix proof-nets) Let L denote an arbitrary terminal link in a MALL+Mix proof-net Θ. If L can be removed then proof-structures obtained after removing L are still MALL+Mix proof-nets.

Theorem 3 (MALL+Mix sequentialization) *A MALL proof-structure Θ is sequentializable for MALL+Mix iff Θ is a MALL+Mix proof-net.*

Proof We prove the if part (since the only if part is straightforwardly checked by the adequacy theorem). We prove this part by double induction on $(n(\Theta), m(\Theta))$, where $n(\Theta)$:= number of $\otimes$-links in Θ and $m(\Theta)$:= number of &-links in Θ.

(**Base Case**:) The case where $n(\Theta) = 0$. In this case Θ is a PALL proof-structure. From Corollary 2 of Section 2, every PALL proof-structure is sequentializable for PALL+Mix.

(**Induction Case**:) The case where $n(\Theta) > 0$:
First of all, we consider the following procedure: From Remark 1, by successive removal of terminal $\{⅋, \oplus_1, \oplus_2\}$-links starting from Θ, we obtain a proof-structure Θ^* without any terminal $\{⅋, \oplus_1, \oplus_2\}$-links. Θ^* is a MALL+Mix proof-net by Remark 9. Note that $(n(\Theta^*), m(\Theta^*)) = (n(\Theta), m(\Theta))$. We shall prove that Θ^* is sequentializable for MALL+Mix, which implies that so is Θ.
(**Case 1**) The case where Θ^* has a terminal &-link: By Remark 1, we can remove the link to obtain two proof-nets Θ_i^* $(i = 1, 2)$ by Remark 9. Since $m(\Theta^*) > m(\Theta_i^*)$ with $n(\Theta^*) \geq n(\Theta_i^*)$, the induction hypothesis implies that for all $i \in \{1, 2\}$ Θ_i^* is MALL+Mix sequentializable, hence so is Θ^*.
(**Case 2**) Negation of Case 1: From softness of MALL proof-structures (Theorem 1), Θ^* has a terminal link, which in this case must be a $\otimes$-link.

From (3) below Definition 15 we have $\Theta^* = \mid\Theta^*\mid \lceil L_1, \ldots, L_n\rceil$. Since the images of $L_1, \ldots, L_n$ are all the $\otimes$-links in Θ^*, we may assume without loss of generality that L_n is a terminal ⅋-link of $|\Theta^*|$, whose image in Θ^* is a terminal $\otimes$-link.

Now consider the proof-structure $\mid\Theta^*\mid \lceil L_1, \ldots, L_{n-1}\rceil$, whose number of $\otimes$-links is less than that of Θ^*. $|\Theta^*| \lceil L_1, \ldots, L_{n-1}\rceil$ is a MALL+Mix proof-net since it is obtained from Θ^* by replacing the image of L_n by a ⅋-link and the acyclicity condition is preserved under the replacement. Hence by I.H., $|\Theta^*| \lceil L_1, \ldots, L_{n-1}\rceil$ is a sequentializable p-s Θ_π for some MALL+Mix proof π. Thus we have

$$\Theta^* = \mid\Theta^*\mid \lceil L_1, \ldots, L_{n-1}, L_n\rceil = (\mid\Theta^*\mid \lceil L_1, \ldots, L_{n-1}\rceil)\lceil L_n\rceil = \Theta_\pi\lceil L_n\rceil$$

Since the MALL+Mix sequentializable p-s Θ_π has a terminal ⅋-link L_n, among possible MALL+Mix proofs π such that $\Theta_\pi = \mid\Theta^*\mid \lceil L_1, \ldots, L_{n-1}\rceil$ we can choose such π whose last inference is a ⅋-inference interpreted by the link L_n. This indicates that Θ^* is a simple proof-structure. Moreover this case guarantees that Θ^* satisfies the conditions (i) and (ii) of Proposition 4 (splitting lemma). Thus the splitting lemma implies that the condition (iii) holds for Θ^*. Hence we can remove a certain $\otimes$-link guaranteed in (iii) to obtain two proof-structures Θ_1^* and Θ_2^*, which are proof-nets by Remark 9. Since $n(\Theta^*) > n(\Theta_1^*)$ and $n(\Theta^*) > n(\Theta_2^*)$ with $m(\Theta^*) = m(\Theta_1^*) + m(\Theta_2^*)$, I.H. implies that $n(\Theta_1^*)$ and $n(\Theta_2^*)$ are MALL+Mix sequentializable, hence so is Θ^*. $\square$

Remark 10 (Restriction to multiplicatives) Our proof of Theorem 3 contains a proof for a MLL+Mix sequentialization theorem (cf. the paragraph at the bottom of Section 4.1).

The following is a direct corollary of Theorem 3 (note that our proof of Theorem 3 does not presume the MALL sequentialization theorem of [9]):

Corollary 5 (MALL sequentialization) *A MALL proof-structure Θ is MALL sequentializable if and only if (i) for every switching S the graph Θ_S is acyclic, and (ii) for every normal switching S_0 the graph Θ_{S_0} is connected.*

Proof First note the following straightforward observation:
(†) If a p-s Θ is sequentializable for MALL+Mix but not for MALL, then there exists a normal switching S_0 such that the graph Θ_{S_0} is not connected.
With (†), the if part of the corollary is direct: Theorem 3 together with (i) implies that Θ is MALL+Mix sequentializable. But (†) together with (ii) implies that Θ is sequentializable without Mix. $\square$

5 Remarks

5.1 Remark on Cycles for MALL+Mix Criterion

In this subsection we shall remark that a class of cycles necessary in our MALL+Mix criterion is wider than that for original MALL criterion.

Definition 19 (local cycle) A *local cycle* in a graph Θ_S is a cycle all of whose nodes occur in a connected component in the graph Θ_{S_0}, where S_0 denotes the *associated normal switching* for S; i.e., S_0 is the normal switching which determines the same valuation and the same choices of left or right (on $⅋$-links) as S.

It is straightforward to weaken the condition of Girard's MALL proof-net (Definition 10 of [9]) into the following form by replacing the notion of cycles with that of *local cycles*:

Lemma 7 (local cycles suffice for MALL criterion) *Θ is a MALL proof-net if and only if for every switching S the graph Θ_S is connected and contains no local cycle.*

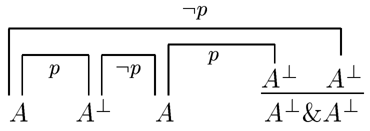
Proof Under the constraint that every switching S yields a connected graph Θ_S, if for some switching S, Θ_S has a cycle then it must be a local cycle. $\square$

As opposed to the MALL case shown in the above lemma, it is impossible to weaken our MALL+Mix correctness criterion (Theorem 3) in the same manner. A counter example is given by the following remark:

For this, first let us consider the MALL+Mix proof given by:

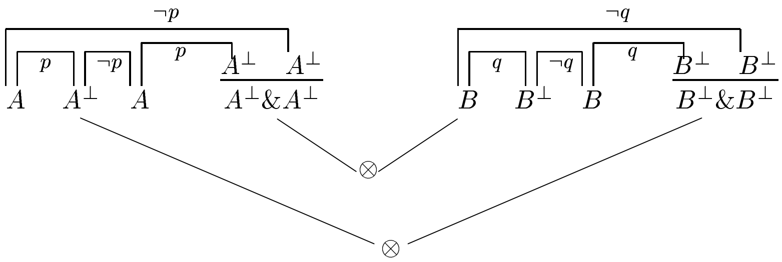
$$\frac{\vdash A, A^{\perp}, A, A^{\perp} \quad \vdash A, A^{\perp}, A, A^{\perp}}{\vdash A, A^{\perp}, A, A^{\perp} \& A^{\perp}} \ (\&)$$

This proof is interpreted by the following MALL proof-structure, where p is an eigenweight associated with the &-link with conclusion $A^{\perp} \& A^{\perp}$:



Then our remark is:

Remark 11 (local cycles do not suffice for MALL+Mix criterion) Let Θ denote the following MALL proof-structure which is obtained from union of two copies of the above proof-structure by drawing $\otimes$-links as shown. Then Θ is not sequentilizable for MALL+Mix. On the other hand for all switching S, Θ_S does not contain any local cycle.



In the above, p (respectively q) is an eigenweight associated with the &-link of conclusion $A^{\perp}\&A^{\perp}$ (respectively $B^{\perp}\&B^{\perp}$).

5.2 Remark on Dependency Condition

Finally in this brief subsection, we shall remark on the status of the *dependency condition* in this paper. The condition is an important issue because sequentialization without the dependency condition has been an open problem (cf. Girard [9]). The problem has arisen because without the restrictions of the dependency condition, the notion of dependency becomes weaker.

Recently D. Hughes and R. van Glabbeek [12] have found that the dependency condition cannot be eliminated in order to retain Girard's original criterion. Based on this, they have announced a stronger correctness criterion than Girard's for their new system of proof-structures without the dependency condition.

Let us note an important remark on the situation of Girard's technical condition *without* the dependency condition. First, as opposed to our result of Section 2, once we remove the constraints of the dependency condition, Girard's technical condition no longer implies softness. This is due to the fact that the equivalent version of the technical condition (cf. Definition 20) used in this paper becomes strictly stronger when the dependency condition is removed. Secondly, however, except for this implicit use of the dependency condition, all our proofs in Section 2 are free from dependency. Hence all the results of Section 2 are still valid without the constraint of the dependency condition, if Girard's technical condition denotes the stronger version.

Since our methods of MALL sequentialization (Theorem 3) in this paper are based on PALL+Mix sequentialization, which is an important consequence of softness, the methods cannot directly apply to Hughes-Glabbeek's system of proof-structures, which are not soft. However we remark that the version of the technical condition naturally works for classifying Hughes-Glabbeek's proof-structures into two classes; one class is with the condition, which consists of all the hereditarily soft proof-structures and the other is without the condition. All the sequentializable proof-structures belong to the former class since the condition is necessary to be sequentializable. We may show directly that all the proof-structures in the latter class fail to satisfy Hughes-Glabbeek's correctness criterion. Hence it is still important future work to accommodate the proof-methods of this paper to the hereditarily soft subclass of Hughes-Glabbeek's proof-structures.

6 Appendix

In this appendix, for the reader's convenience, we put the definitions of MALL proof-structures and switchings for them, which yield graphs.

Following Girard's original [9], nice presentations of MALL proof-structures can be found in Abramsky-Melliès [1] and in Laurent [14]:

Definition 20 (MALL proof-structure (cf. [9, 1, 14]) A *proof-structure* Θ consists of the following:

- Occurrences of formulas. Occurrences of links, each of which takes its premise(s) and conclusion(s) among occurrences of formulas and satisfies (i) below.
- A set of *eigenweights* $\{p_{L_1}, \ldots, p_{L_n}\}$ where $L_1, \ldots, L_n$ is the list of all &-links occurring in Θ and each p_{L_i} is a boolean variable associated with a & link L_i.
- For each occurrence A of a formula and occurrence L of a link, a *weight* $w(A)$ and a *weight* $w(L)$, each of which is in the non-zero boolean algebra generated by the eigenweights and satisfies (ii),(iii) and (iv) below:

link L	(i) $L\dfrac{\text{premise(s)}}{\text{conclusion(s)}}$:	(ii) weights of L and its premise(s):
axiom-link	$\overline{\quad\ } \atop A \quad A^{\perp}$	
$\otimes$-link	$\dfrac{A \quad B}{A \otimes B}$	$w(L) = w(A) = w(B)$
$⅋$ -link	$\dfrac{A \quad B}{A ⅋ B}$	$w(L) = w(A) = w(B)$
&-link	$\dfrac{A \quad B}{A \& B}$	$w(A) = p_L.w(L)$ and $w(B) = \neg p_L.w(L)$
$\oplus_1$-link	$\dfrac{A}{A \oplus B}$	$w(L) = w(A)$
$\oplus_2$-link	$\dfrac{B}{A \oplus B}$	$w(L) = w(B)$

(iii) $w(A) = \Sigma w(L)$ with L ranging over the links whose conclusion is A. Moreover the sum satisfies the *disjointness property*; i.e., if L_1 and L_2 are distinct links sharing the same conclusion A then $w(L_1).w(L_2) = 0$.
(iv) $w(A) = 1$ for a formula A which is not a premise of any link, i.e. which is a *conclusion* of Θ.

Moreover a proof-structure Θ satisfies the following two conditions, referred to as the *dependency* and the *technical* conditions:

dependency condition: Every weight of a formula and a link in Θ is a product of eigenweights and negations of eigenweights; i.e., every weight is a monomial.

Here we make the convention that all monomial weights to be considered are reduced in that occurrences of $\epsilon p.\epsilon p$ are replaced by ϵp and occurrences of $p.\neg p$ are replaced by 0. Under the convention a (reduced) monomial weight w is defined to *depend* on an eigewnweight p if $\epsilon.p$ appears in w with $\epsilon \in \{1, \neg\}$.

technical condition: For every &-link L with its associated eigenweight p_L;

1. If a weight w occurring in Θ depends on p_L, then $w \subseteq w(L)$.
2. $w(L)$ does not depend on p_L.

These 1 and 2 are equivalent to the following 1':

1'. If a weight w occurring in Θ depends on p_L, then $w \subset w(L)$.

For monomial weights w, this definition of technical condition is equivalent to Girard's original definition:

- For any element w of the boolean algebra generated by the weights occurring in Θ, $w.\neg w(L)$ belongs to the boolean algebra generated by the eigenweights distinct from p_L.

Definition 21 (Switching S and Graph Θ_S of MALL Proof Structures Θ (cf. [9, 1]))

- A *switching* S of a proof-structure Θ consists of the following three choices:

  (i) The choice of a *valuation* φ_S, which is a function from the set $\{p_{L_1}, \dots, p_{L_n}\}$ of eigenweights to $\{0,1\}$. φ_S induces a function from the weights of Θ to $\{0,1\}$. The *slice* $\varphi_S(\Theta)$ is obtained by restricting Θ to the formulas A such that $\varphi_S(w(A)) = 1$.

  (ii) For each $⅋$-link L of $\varphi_S(\Theta)$, a choice $S(L) \in \{l, r\}$.

  (iii) For each &-link L of $\varphi_S(\Theta)$, a choice of a formula $S(L)$, called a *jump of* L, so that $S(L)$ is a conclusion of a link whose weight depends on p_L. A jump is *normal* if $S(L)$ is the premise A of L such that $\varphi_S(w(A)) = 1$. A *proper jump* is a jump which is not normal.

- A *normal switching* is a switching with no proper jump.

- For a switching S of a proof-structure Θ, the *graph* Θ_S is drawn as follows:

  - The vertices of Θ_S are the occurrences of the formulas of $\varphi_S(\Theta)$.

  - For all axiom-links of $\varphi_S(\Theta)$, we draw an edge between its conclusions.

  - For all $\oplus_i$-links of $\varphi_S(\Theta)$, we drawn an edge between the conclusion and the premise.

  - For all $\otimes$-links of $\varphi_S(\Theta)$, we drawn an edge between the conclusion and the left premise, and between the conclusion and the right premise.

  - For all $⅋$-links of $\varphi_S(\Theta)$, we drawn an edge between the conclusion and the premise (left or right) selected by $S(L)$.

  - For all &-links of $\varphi_S(\Theta)$, we drawn an edge between the conclusion and the jump $S(L)$ selected by S.

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