KBO Orientability

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Term Rewriting

Definition

- pair of terms $l \to r$ is rewrite rule if $l \notin \mathcal{V} \land \mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(l)$
- term rewrite system (TRS) is set of rewrite rules
- (rewrite relation) $s \to_{\mathcal{R}} t$ if $\exists l \to r \in \mathcal{R}$, context C, substitution σ. $s = C[l\sigma] \;\land\; t = C[r\sigma]$

Example

TRS $\mathcal{R}$

$$x + \mathsf{0} \to x \qquad\qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times \mathsf{0} \to \mathsf{0} \qquad\qquad x \times \mathsf{s}(y) \to x \times y + x$$

rewriting

$$\begin{aligned} \mathsf{s}(\mathsf{0}) \times \mathsf{s}(\mathsf{0}) &\to_{\mathcal{R}} \mathsf{s}(\mathsf{0}) \times \mathsf{0} + \mathsf{s}(\mathsf{0}) \\ &\to_{\mathcal{R}} \mathsf{0} + \mathsf{s}(\mathsf{0}) \\ &\to_{\mathcal{R}} \mathsf{s}(\mathsf{0} + \mathsf{0}) \\ &\to_{\mathcal{R}} \mathsf{s}(\mathsf{0}) \end{aligned}$$

terminated

Termination

DEFINITION

TRS $\mathcal{R}$ is terminating if there is no infinite rewrite sequence
$t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$

QUESTION

how to prove termination?

☞ Knuth-Bendix order (KBO)

- introduced by Knuth and Bendix, 1970
- best studied termination methods
- great success in theorem provers (Waldmeister, Vampire, ...)

Knuth-Bendix Orders

Definition

- precedence $>$ is proper order on function symbols $\mathcal{F}$
- weight function (w, w_0) is pair in ${\mathbb{R}_{\geqslant 0}}^{\mathcal{F}} \times \mathbb{R}_{\geqslant 0}$
- weight of term t is

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathcal{V} \\ w(f) + w(t_1) + \cdots + w(t_n) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- weight function (w, w_0) is admissible for precedence $>$ if

$$w(f) > 0 \qquad \text{or} \qquad f \geqslant g$$

for all unary functions f and all functions g

Definition

Knuth-Bendix order $>_{\mathrm{kbo}}$ on terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$:
$s >_{\mathrm{kbo}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either

- $w(s) > w(t)$, or
- $w(s) = w(t)$ and
  - $s = f^n(t)$ and $t \in \mathcal{V}$ for some unary f and $n \geqslant 1$; or
  - $s = f(s_1, \ldots, s_{i-1}, s_i, \ldots, s_n)$, $t = f(s_1, \ldots, s_{i-1}, t_i, \ldots, t_n)$, and $s_i >_{\mathrm{kbo}} t_i$; or
  - $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f > g$

Definition

let $X \subseteq \mathbb{R}_{\geqslant 0}$. TRS $\mathcal{R}$ is KBO_X terminating if

- $\exists$ precedence $>$
- $\exists$ admissible weight function $(w, w_0) \in X^{\mathcal{F}} \times X$

such that $l >_{\mathrm{kbo}} r$ for all $l \to r \in \mathcal{R}$

Theorem Knuth and Bendix, 1970

TRS is terminating if it is $\mathrm{KBO}_{\mathbb{N}}$ terminating

Theorem Knuth and Bendix, 1970

TRS is terminating if it is $\mathit{KBO}_{\mathbb{N}}$ *terminating*

Theorem Dershowitz, 1979

TRS is terminating if it is $\mathit{KBO}_{\mathbb{R}_{\geqslant 0}}$ *terminating*

Theorem Dick, Kalmus, and Martin, 1990

$\mathit{KBO}_{\mathbb{R}_{\geqslant 0}}$ *termination is decidable*

Theorem Korovin and Voronkov, 2001, 2003

- *TRS is* $\mathit{KBO}_{\mathbb{N}}$ *terminating* $\iff$ *it is* $\mathit{KBO}_{\mathbb{R}_{\geqslant 0}}$ *terminating*
- $\mathit{KBO}_{\mathbb{R}_{\geqslant 0}}$ *termination is decidable within polynomial time*

Theorem Zankl and Middeldorp, 2007

$\mathit{KBO}_{\{0,1,\ldots,B\}}$ *termination* ($B \in \mathbb{N}$) *can reduce to SAT and PBC*

Main Result

THEOREM

$\mathcal{R}$ is KBO$_{\mathbb{N}}$ terminating $\iff$ *$\mathcal{R}$ is KBO$_{\{0,1,\ldots,B\}}$ terminating*
where, $B = n^{2^{n+1}}$

COROLLARY

Zankl and Middeldorp's SAT and PBC encodings are complete for this B

Summary

let $\mathcal{R}$ be TRS of size $n = \sum_{l \to r \in \mathcal{R}} (|l| + |r|)$ and $B = n^{2^{n+1}}$

$\mathcal{R}$ is $\text{KBO}_{\mathbb{R}_{\geqslant 0}}$ terminating

$\iff$ $\mathcal{R}$ is $\text{KBO}_{\mathbb{N}}$ terminating — Korovin and Voronkov

$\iff$ $\mathcal{R}$ is $\text{KBO}_{\{0,1,\ldots,B\}}$ terminating — this talk

- theoretical interest of decidability issue is more or less closed

FUTURE WORK

find optimal B