Satisfiability in monadic Gödel logics

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History

- Gödel (1933) – finitely valued logics
- Dummett (1959) – infinitely valued propositional Gödel logics
- Horn (1969) – linearly ordered Heyting algebras
- Takeuti-Titani (1984) – intuitionistic fuzzy logic
- Avron (1991) – hypersequent calculus
- Hájek (1998) – t-norm based logics
- Viennese group (Baaz, Beckmann, Ciabattoni, Fermüller, Goldstern, Veith, Zach, P.) (since 90ies) – proof theory, counting, Kripke, quantified propositional, (monadic) fragments, ...

Syntax and Semantics

Usual first-order language, $\neg A$ is defined as $A \to \bot$.

Evaluations

Fix a truth value set $\{0, 1\} \subseteq V \subseteq [0, 1]$

$$\mathcal{I} : \text{Atom} \mapsto V$$

maps atomic formulas to elements of V.

Syntax and Semantics cont.

Extension of $\mathcal{I}$ to all formulas:

$$\mathcal{I}(A \wedge B) = \min\{\mathcal{I}(A), \mathcal{I}(B)\}$$

$$\mathcal{I}(A \vee B) = \max\{\mathcal{I}(A), \mathcal{I}(B)\}$$

$$\mathcal{I}(A \to B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B) \\ 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B) \end{cases}$$

$$\mathcal{I}(\forall x A(x)) = \inf\{\mathcal{I}(A(u)) : u \in U\}$$

$$\mathcal{I}(\exists x A(x)) = \sup\{\mathcal{I}(A(u)) : u \in U\}$$

Validity and Satisfiability

validity (logic)	$\mathbf{G}_V^{(\Delta)}$	$A : \forall \mathcal{I} : \mathcal{I}(A) = 1$
p-satisfiability	p-$\mathbf{SAT}$-$\mathbf{G}_V^{(\Delta)}$	$A : \exists \mathcal{I} : \mathcal{I}(A) \geq p$
1-satisfiability	1-$\mathbf{SAT}$-$\mathbf{G}_V^{(\Delta)}$	$A : \exists \mathcal{I} : \mathcal{I}(A) = 1$

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Remark

Different V might generate the same set of formulas.

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Warning

Satisfiability and Validity are *not* dual in the many-valued setting!

Descriptive Set Theory

Cantor-Bendixon Derivatives and Ranks

Polish spaces, i.e. separable, completely metrizable topological spaces. $\mathbb{R}$ is a Polish space.

$$X' = \{x \in X : x \text{ is limit point of } X\}$$

Theorem (Cantor-Bendixon)

Let X be a polish space. For some countable ordinal α_0, $X^\alpha = X^{\alpha_0}$ for all $\alpha \geq \alpha_0$ (X^{α_0} is the perfect kernel).

CB Ranks for countable closed sets

- If X is countable, then $X^\infty = \emptyset$.
  (every perfect set has at least cardinality of the continuum)

CB Ranks for countable closed sets

- If X is countable, then $X^\infty = \emptyset$.
  (every perfect set has at least cardinality of the continuum)
- rank of an element: $|x|_{\mathrm{CB}} = \sup\{\alpha : x \in X^\alpha\}$
- rank of X: $|X|_{\mathrm{CB}} = \sup\{|x|_{\mathrm{CB}} : x \in X\}$

Some results for validity

(recursive) Axiomatizability of $\mathbf{G}_V$

- V uncountable, $0 \in V^{\infty}$: yes
- V uncountable, $|0|_{\mathrm{CB}} = 0$: yes
- otherwise: not r.e.

Decidability of monadic fragment

all are undecidable but one open case:
$V_{\uparrow} = \{1 - 1/n\} \cup \{1\}$

Results for SAT

Monadic logics

- $|0|_{CB} = 0$: decidable
  (subclasses: finite, prenex, ∃-fragment, monadic witnessed)
- $|0|_{CB} \geq 1$, 3 predicate symbols one of which is constant interpreted strictly between 0 and 1: undecidable
- $|0|_{CB} \geq 2$, 3 predicate symbols: undecidable
- $|0|_{CB} = 1$, no special predicate constant: open

Results for sat cont.

Monadic with Δ

finite V is decidable, otherwise undecidable

Subclass S1Δ

Decidable, only two logics: $|1|_{CB} = 0$ and $|1|_{CB} > 0$

Subclass S1$\Delta\sim$ (with involutive negation)

Same as without $\sim$

Monadic logics: $|0|_{CB} = 0$

Theorem

$$\textbf{SAT-G}_V = \textbf{SAT-CL}$$

Proof

If $A \in$ **SAT-CL**, then it is also in $\textbf{SAT-G}_V$ since $\{0, 1\} \subseteq V$.
If $A \in \textbf{SAT-G}_V$, define $\mathcal{I}_{CE}$ as follows:

$$\mathcal{I}_{CE}(A) = \begin{cases} 1 & \mathcal{I}_G(A) > 0 \\ 0 & \text{o.w.} \end{cases}$$

Induction on formulas, critical case if $\forall x A(x)$ with $\mathcal{I}_G(\forall x A(x)) = 0$, but since 0 is isolated, there is a witness $\mathcal{I}_G(A(u)) = 0$.

Consequences for 0 isolated

The following fragments are decidable due to the decidability of **SAT-G**$_V$ for 0 isolated in V:

- finitely valued logics
- prenex fragment
- ∃-fragment
- monadic witnessed

Remark

All these satisfiability logics coincide with **SAT-CL** (for the resp. fragments)

INTERLUDE: V INFINITE, $\mathbf{G}_V^\Delta$

Evaluation of Δ

$$\mathcal{I}(\Delta A) = \begin{cases} 1 & \mathcal{I}(A) = 1 \\ 0 & \text{otherwise} \end{cases}$$

The definition of Δ parallels the (computed) evaluation of $\neg A$:

$$\mathcal{I}(\neg A) = \begin{cases} 1 & \mathcal{I}(A) = 0 \\ 0 & \text{otherwise} \end{cases}$$

UNDECIDABILITY OF **SAT-G$_V^\Delta$**

Logic CE

Classical theory CE of two equivalence relations.

$$A = \mathcal{Q}^* \bigvee_j \bigwedge_k (x_j^k \equiv_i y_j^k)^l$$

Fact

SAT-CE is not even recursively enumerable

Theorem

CE can be faithfully interpreted in monadic $\mathbf{G}_V^\Delta$, and thus monadic **SAT-G$_V^\Delta$** is undecidable.

Interpreting CE in $\mathbf{G}_V^\Delta$

Proof

$$\sigma(x \equiv_i y) = \Delta(P_i x \leftrightarrow P_i y)$$

$$\lambda \text{ injective } \{[u]_i : u \in \mathcal{U}_{\mathrm{CE}}, i = 1, 2\} \to V \setminus \{0, 1\}$$

$$\mathcal{I}_{\mathrm{G}}(P_i u) = \lambda([u]_i)$$

$|0|_{\mathrm{CB}} \geq 1$, THREE PREDICATE SYMBOLS

Theorem

If $|0|_{\mathrm{CB}} \geq 1$ in V, there are at least three predicate symbols, one of which is constant strictly between 0 and 1, then **SAT-G**$_V$ is undecidable.

Proof

As above, but we have to translate negation, too

$$\sigma(x \equiv_i y) = (P_i x \leftrightarrow P_i y)$$

$$\sigma(x \not\equiv_i y) = (P_i x \leftrightarrow P_i y) \to S$$

$$\lambda \text{ injective } \{[u]_i : u \in \mathcal{U}_{\mathrm{CE}}, i = 1, 2\} \to V \cap (0, \mathcal{I}_{\mathrm{G}}(S))$$

$$\mathcal{I}_{\mathrm{G}}(P_i u) = \lambda([u]_i)$$

$|0|_{CB} \geq 2$, THREE PREDICATE SYMBOLS

Theorem

If $|0|_{CB} \geq 2$ in V and there are at least three predicate symbols, then $\mathbf{SAT\text{-}G}_V$ is undecidable.

Proof ideas

- forcing third predicate to decrease to 0:
  $\neg \forall x S(x) \wedge \forall x \neg\neg S(x)$
- confine interpretations to intervals below $S(u)$
- parallel execution of the above construction for each of these intervals
- multiplication of the universe for each of these intervals

The translation

$$\sigma_{a,b}(\forall r B) = \forall r(P_1 r \prec Pb \vee Pa \prec P_1 r \vee P_2 r \prec Pb \vee Pa \prec P_2 r \vee \sigma_{a,b}(B)$$

$$\sigma_{a,b}(\exists r B) = \exists r((Pb \prec P_1 r \prec Pa) \wedge (Pb \prec P_2 r \prec Pa) \wedge \sigma_{a,b}(B))$$

$$\sigma_{a,b}(\bigvee_j \bigwedge_k (r_j^k \equiv_i s_j^k)^l) = \bigvee_j \bigwedge_k \sigma((r_j^k \equiv_i s_j^k)^l)$$

$$\sigma_{a,b}(r \equiv_i s) = (P_i r \leftrightarrow P_i s)$$

$$\sigma_{a,b}(r \not\equiv_i s) = ((P_i r \leftrightarrow P_i s) \to Pa))$$

$$\begin{aligned}\tau(A) = \neg\forall x Px \wedge \forall x \neg\neg Px \wedge \\ \forall x(Px \vee \exists y \exists z[&Pz \prec Py \wedge Py \prec Px \wedge \\ &\forall u(Pu \to Pz \vee Py \to Pu) \wedge \\ &\exists w(Pz \prec P_1 w \prec Py \wedge Pz \prec P_2 w \prec Py) \wedge \\ &\sigma_{y,z}(A)])\end{aligned}$$

The open case

That leaves the case that $|0|_{\mathrm{CB}} = 1$ with no constant predicate symbol open.

Lemma

If $|0|_{\mathrm{CB}} = 1$ in V, then $\textbf{SAT-G}_V = \textbf{SAT-G}_{V_\downarrow}$ where $V_\downarrow = \{1/n : n \in \mathbb{N}\} \cup \{0\}$

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Remark

Remember that the only open case for validity is $V_\uparrow$.

Summary for monadic logics

- $|0|_{CB} = 0$: $\mathbf{SAT\text{-}G}_V$ decidable (subclasses: finite, prenex, ∃-fragment, monadic witnessed)
- $|0|_{CB} \geq 1$, 3 predicate symbols one of which is constant interpreted strictly between 0 and 1: $\mathbf{SAT\text{-}G}_V$ undecidable
- $|0|_{CB} \geq 2$, 3 predicate symbols: $\mathbf{SAT\text{-}G}_V$ undecidable
- $|0|_{CB} = 1$, no special predicate constant: $\mathbf{SAT\text{-}G}_V$ open
- finite V: $\mathbf{SAT\text{-}G}_V^{\Delta}$ decidable
- infinite V: $\mathbf{SAT\text{-}G}_V^{\Delta}$ undecidable

Where to go from here?

The fragment S1Δ

Definition

The fragment S1Δ consists of all formulas in the language with Δ of the form

$$\bigvee_{i=1}^{n} (\exists x A_1^i(x) \wedge \ldots \wedge \exists x A_{n_i}^i(x) \wedge \forall x B_1^i(x) \wedge \ldots \wedge \forall x B_{m_i}^i(x))$$

where A_k^i and B_k^i quantifier-free containing no constant symbols.

Background

Medical database of the General Hospital in Vienna, development of an expert system for medical decisions

Results for S1Δ

- $|1|_{CB} = 0$ in V, then **SAT-**S1Δ is decidable
- $|1|_{CB} > 0$ in V, then **SAT-**S1Δ is decidable
- the above two cases are the only ones, and they are different (the set of satisfiable formulas are different)
- adding the involutive negation $\sim$ does not change the status

THE CASE $|1|_{CB} > 0$ (THE BAD ONE)

Δ-chains

Let $P \prec Q$ stand for $\neg\Delta(Q \to P)$

Let $P \geq Q$ stand for $\Delta(P \to Q) \wedge \Delta(Q \to P)$.

Let F be any formula in S1Δ and $A_1, \ldots, A_n$ be the predicates occurring in F. A Δ-*chain* over F is any formula of the form

$$(\bot \bowtie_0 A_{\pi(1)}(x)) \wedge (A_{\pi(1)}(x) \bowtie_1 A_{\pi(2)}(x)) \wedge (A_{\pi(n)}(x) \bowtie_n \top)$$

where π is a permutation of $\{1, \ldots, n\}$, $\bowtie_i$ is either $\prec$ or $\geq$, and at least one of the $\bowtie_i$'s stands for $\prec$.

CHAINS CONT.

- every Δ-chain describes a possible ordering of the values of predicates of F
- every Δ-chain C_i induces equivalence classes over the predicates of F
- if C_F is the set of all chains, then $\bigvee_{C \in C_F} C$ is a tautology in $\mathbf{G}_V^\Delta$.

Chains cont.

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- if C_F is the set of all chains, then $\bigvee_{C \in C_F} C$ is a tautology in $\mathbf{G}_V^\Delta$.

Syntactic evaluation

For every quantifier-free subformula $A(x)$ of F and every Δ-chain C there is a predicate symbol (or $\top$ or $\bot$) $P^C_{A(x)}$ such that

$$\mathcal{I}(C \wedge A(x)) = \mathcal{I}(C \wedge P^C_{A(x)})$$

Reduction of the existential quantifier

$$\begin{aligned} \exists x A(x) &\stackrel{\text{SAT}}{\equiv} \exists x((\bigvee_{C \in \mathcal{C}_F} C) \wedge A(x)) \\ &\stackrel{\text{SAT}}{\equiv} \bigvee_{C \in \mathcal{C}_F} \exists x(C \wedge A(x)) \\ &\stackrel{\text{SAT}}{\equiv} \bigvee_{C \in \mathcal{C}_F} \exists x(C \wedge P^C_{A(x)}) \end{aligned}$$

- delete disjuncts with $P^C_{A(x)}$ being $\bot$
- if in a disjunct $P^C_{A(x)}$ is equal to $\top$ then the formula is already satisfiable
- collect the remaining chains in Γ

Reduction of the universal quantifier

$$
\begin{aligned}
\forall x B(x) &\overset{\text{SAT}}{\equiv} \Delta \forall x B(x) \overset{\text{SAT}}{\equiv} \forall x \, \Delta B(x) \\
&\overset{\text{SAT}}{\equiv} \forall x((\bigvee_{C \in \mathcal{C}_F} C) \wedge \Delta B(x)) \overset{\text{SAT}}{\equiv} \forall x(\bigvee_{C \in \mathcal{C}_F} (C \wedge \Delta B(x))) \\
&\overset{\text{SAT}}{\equiv} \forall x(\bigvee_{C \in \mathcal{C}_F} (C \wedge P^{C}_{\Delta B(x)})) \\
&\overset{\text{SAT}}{\equiv} \forall x(\bigvee_{C \in \mathcal{C}' \subseteq \mathcal{C}_F} C) \\
&\overset{\text{SAT}}{\equiv} \forall x \bigwedge_j \bigvee_k \mathcal{O}_{j,k} \overset{\text{SAT}}{\equiv} \bigwedge_j \forall x \bigvee_k \mathcal{O}_{j,k} \\
&\overset{\text{SAT}}{\equiv} \bigwedge_j \forall x \Pi_j
\end{aligned}
$$

Satisfiability condition

$$F \overset{\text{SAT}}{\equiv} \bigvee_{C \in \Gamma} \exists x (C \wedge P^{C}_{A(x)}) \wedge \bigwedge_{j} \forall x \Pi_j$$

The formula F is satisfiable iff there is a Δ-chain C in Γ such that C is compatible with each Π_i.

Note

Both Γ and Π_i are finite sets, so this is a finite check

Construction of the model (crucial part)

$$F \overset{\text{SAT}}{\equiv} \bigvee_{C \in \Gamma} \exists x (C \wedge P^{C}_{A(x)}) \wedge \bigwedge_{j} \forall x \Pi_j$$

Construction

- we have to ensure that the evaluation of the existential quantifier above actually takes the value 1
- take as universe of objects the natural numbers
- evaluations of atomic formulas (but those from the equivalence class of $\perp$ have 1 as limit (not isolated) with respect to the objects
- since 1 is not isolated the chain of equivalence classes can be ‘compressed’ to 1

Construction of the model (crucial part)

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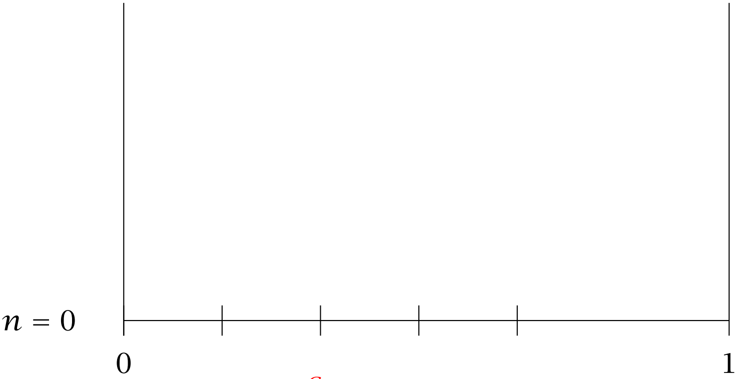

$$[\bot] \prec [P_{i_1}] \prec [P^{C}_{A(x)}] \prec [P_{i_3}] \prec [P_{i_4}] \prec \cdots \prec [\top]$$

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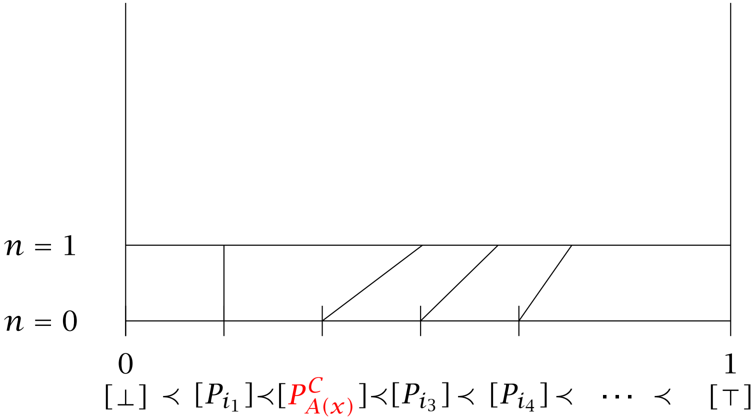

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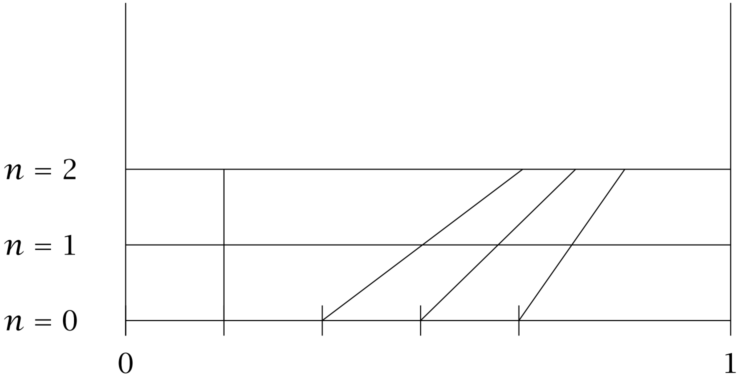

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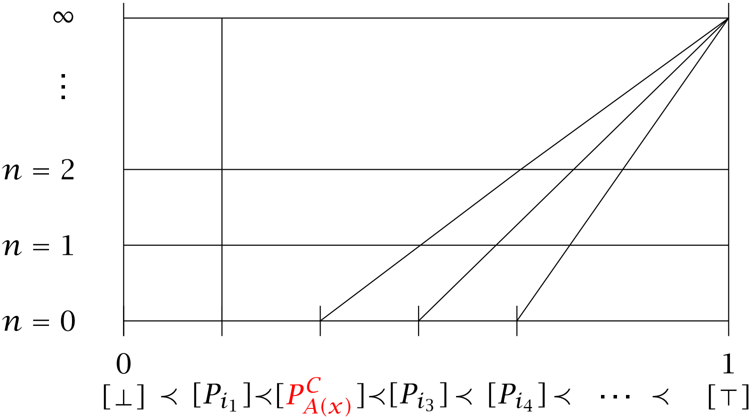

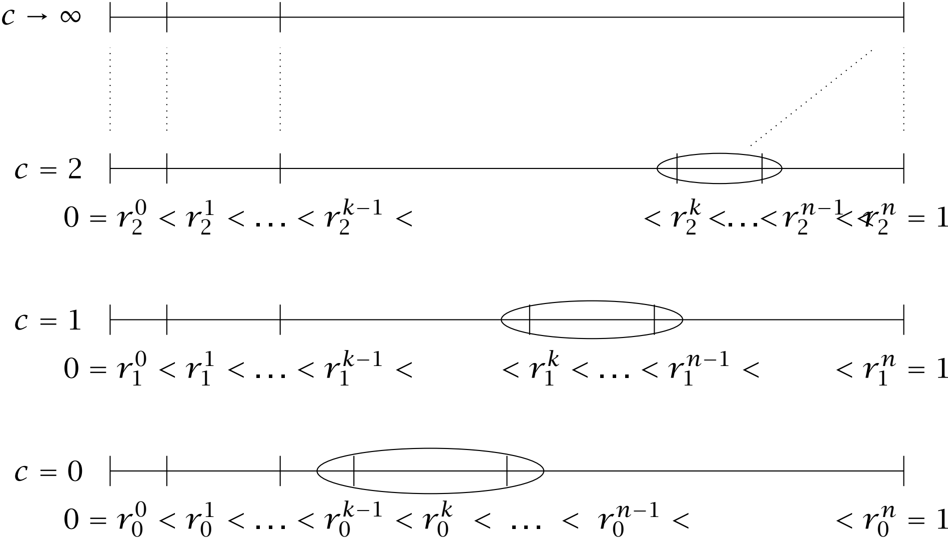
$c \to \infty$
$c = 2$
$0 = r_2^0 < r_2^1 < \dots < r_2^{k-1} <$
$< r_2^k < \dots < r_2^{n-1} < < r_2^n = 1$
$c = 1$
$0 = r_1^0 < r_1^1 < \dots < r_1^{k-1} <$
$< r_1^k < \dots < r_1^{n-1} <$
$< r_1^n = 1$
$c = 0$
$0 = r_0^0 < r_0^1 < \dots < r_0^{k-1} < r_0^k < \dots < r_0^{n-1} <$
$< r_0^n = 1$

The case $|1|_{CB} = 0$

Lemma

A formula A of $\mathsf{S1}\Delta$ is in $\mathbf{SAT\text{-}G}_V^{\Delta}$ if it is in $\mathbf{SAT\text{-}G}_n^{\Delta}$ for $n \geq$ the number of predicates appearing in A plus 2.

Theorem

If $|1|_{CB} = 0$ in V, then $\mathbf{SAT\text{-}G}_V^{\Delta}$ is decidable for $\mathsf{S1}\Delta$.

The involutive negation ~

- restriction on symmetric truth value sets
- extension to specific chains which are symmetric
- satisfiability condition extended by a clause that the syntactic evaluation is in an equivalence class above 1/2

Reduction to propositional satisfiability

The *propositional reduct* A^p of A is defined as follows:

$$(\forall x A)^p = A^p \qquad (\exists x A)^p = A^p$$
$$(A * B)^p = A^p * B^p \text{ for } * \in \{\wedge, \vee, \rightarrow\}$$
$$(\Delta A)^p = \Delta A^p \qquad P_i(\bar{t})^p = P_i$$
$$0^p = 0 \qquad 1^p = 1$$

Reduction cont.

Let

$$F = \forall x A_1(x) \wedge \ldots \wedge \forall x A_m(x) \wedge$$
$$\exists x B_1(x) \wedge \ldots \wedge \exists x B_n(x)$$

and $A = \forall x\, \Delta(A_1(x) \wedge \ldots \wedge A_n(x))$.

Then we have
(i) if V is infinite and 1 isolated,

$$F \in \mathbf{SAT\text{-}G}_V^{\Delta} \quad \leftrightarrow \quad A^p \wedge (\exists x B_1(x))^p \in \mathbf{SAT\text{-}G}_{\infty}^{\Delta} \text{ AND } \ldots \text{ AND}$$
$$A^p \wedge (\exists x B_n(x))^p \in \mathbf{SAT\text{-}G}_{\infty}^{\Delta}$$

Reducion cont.

(ii) if V is infinite, but 1 not isolated, we have

$$F \in \textbf{SAT-G}_V^\Delta \quad \leftrightarrow \quad A^p \wedge \neg\neg(\exists x B_1(x))^p \in \textbf{SAT-G}_\infty^\Delta \text{ AND } \ldots \text{ AND } A^p \wedge \neg\neg(\exists x B_n(x))^p \in \textbf{SAT-G}_\infty^\Delta$$

(iii) Moreover, in case 2. we have: if

$$A^p \wedge (\exists x B_i(x))^p \notin \textbf{SAT-G}_\infty^\Delta \text{ for some } i = 1, \ldots, n.$$

then F does not satisfy the final model property.

Remarks, Conclusions, Questions

- although the satisfiability condition is a finite check, the actual model constructed will not be finite, which in fact is impossible, consider e.g. $F = \forall x \Delta \neg A(x) \wedge \exists x A(x)$
- 1-satisfiability of S1Δ formulas with Δ (same with $\sim$) is NP-complete
- in S1Δ there are only two different logics, distinguished by the property that 1 is isolated or not
- as soon as we consider the 1-variable class with Δ there are countably many different satisfiability logics
- in the absence of Δ not much is known, as many cases will collapse (e.g. $\mathbf{SAT\text{-}G}_V$ for $V = \{0, 1\}$, $V = V_\uparrow$, $V = [0, 1]$).

Time for dinner