Observation of the definitions of the classes…

Def: Class $\mathcal{P}$ (Chapter 5)

Set L is in the class $\mathcal{P} \Leftrightarrow$

There exists a poly-time computable predicate R such that

for each $x \in \Sigma^*$, $x \in L \Leftrightarrow R(x)$

Def: Class $\mathcal{NP}$ (Def 5.2)

Set L is in the class $\mathcal{NP} \Leftrightarrow$

There exists a poly q and a poly-time computable pred. R s.t.

for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \exists\, w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

Def: Class co-$\mathcal{NP}$ (Theorem 5.5)

Set L is in the class co-$\mathcal{NP} \Leftrightarrow$

There exists a poly q and a poly-time computable pred. R s.t.

for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \forall\, w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

計算量クラス間の定義を概観すると…

クラス$\mathcal{P}$の定義(5章)
集合Lがクラス$\mathcal{P}$に入る ⇔
以下を満たす多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow R(x)$

クラス$\mathcal{NP}$の定義(定義5.2)
集合Lがクラス$\mathcal{NP}$に入る ⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \exists w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

クラスco-$\mathcal{NP}$の定義(定理5.5)
集合Lがクラスco-$\mathcal{NP}$に入る ⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \forall w \in \Sigma^* : |w| \leqq q(|x|)[R(x,w)]$

Theorem 5.9

(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

(2) co-$\mathcal{NP} \subseteq \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$

Contraposition: $\mathcal{P} = \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

If we assume $\mathcal{P}=\mathcal{NP}$, for any L we have

$L \in \mathcal{NP} \leftrightarrow \overline{L} \in \mathcal{P}$ ($\mathcal{P} = \mathcal{NP}$)

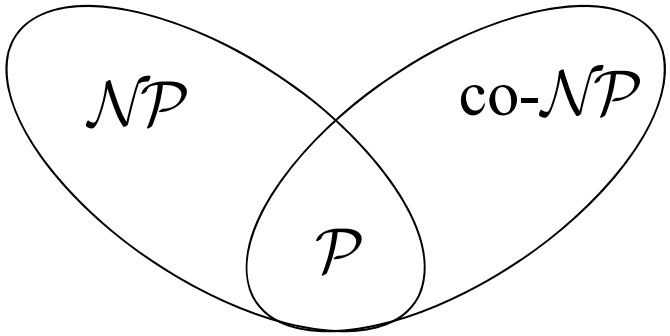
$\leftrightarrow \overline{L} \in \mathcal{P}$ (Exercise 5.5)

$\leftrightarrow \overline{L} \in \overline{\mathcal{NP}}$ ($\mathcal{P} = \mathcal{NP}$)

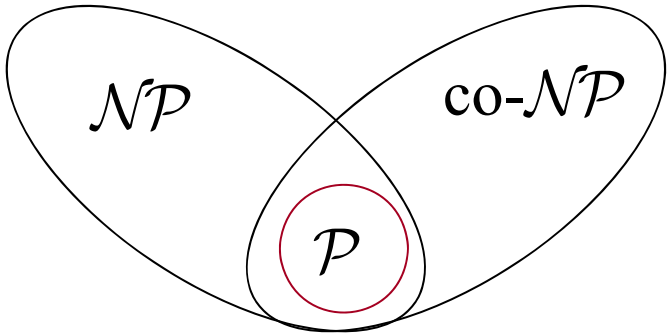
$\leftrightarrow L\ (= \overline{\overline{L}}) \in$ co-$\mathcal{NP}$ (Definition 5.3)

$\therefore \mathcal{NP} =$ co-$\mathcal{NP}$ Q.E.D.

If $\mathcal{NP} \neq$ co-$\mathcal{NP}$ is true,



or



定理5.9.

(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

(2) co-$\mathcal{NP} \subseteq \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

(3) $\mathcal{NP} \neq$ co-$\mathcal{NP} \rightarrow \mathcal{P} \neq \mathcal{NP}$.

対偶: $\mathcal{P} = \mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

$\mathcal{P}=\mathcal{NP}$と仮定すると，すべてのLに対し

$L \in \mathcal{NP} \leftrightarrow \overline{L} \in \mathcal{P}$ （$\mathcal{P} = \mathcal{NP}$ より）

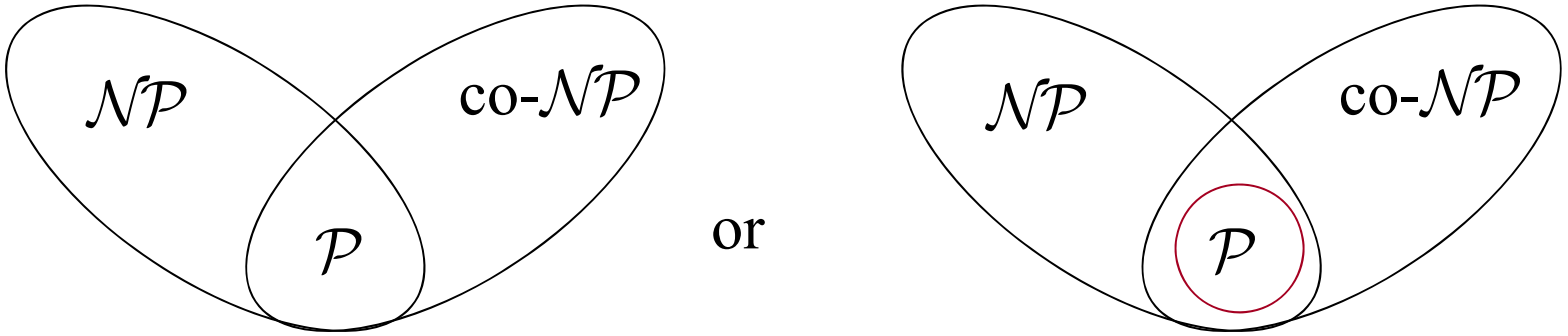
$\leftrightarrow \overline{L} \in \mathcal{P}$ (演習問題5.5)

$\leftrightarrow \overline{L} \in \overline{\mathcal{NP}}$ （$\mathcal{P} = \mathcal{NP}$ より）

$\leftrightarrow L\ (= \overline{L}) \in$ co-$\mathcal{NP}$ (定義5.3より)

$\therefore \mathcal{NP} =$ co-$\mathcal{NP}$ 証明終

$\mathcal{NP} \neq$ co-$\mathcal{NP}$が正しいと



Chapter 6. Analysis on Polynomial-Time Computability

6.1. Polynomial-time Reducibility

Def.6.1:

Let A and B be arbitrary sets.

(1) function h: $A \rightarrow B$: polynomial-time reduction

$\Leftrightarrow$
- (a) h is a total function from Σ^* onto Σ^*
- (b) $x \in \Sigma^* [x \in A \leftrightarrow h(x) \in B]$
- (c) h is polynomial-time computable.

(2) When there is a polynomial-time reduction from A to B, we say A is polynomial-time reducible to B.

Then, we denote by $A \leq_m^P B$

第6章 多項式時間計算可能性の分析

6.1. 多項式時間還元可能性

定義6.1:

AとBを任意の集合とする.

(1) 関数 h: $A \to B$: 多項式時間還元(polynomial-time reduction)

$\Leftrightarrow$
- (a) h はΣ^*からΣ^*への全域的関数
- (b) $x \in \Sigma^* [x \in A \leftrightarrow h(x) \in B]$
- (c) h は多項式時間計算可能.

(2) AからBへの多項式時間還元が存在するとき,
AはBへ多項式時間還元可能という(polynomial time reducible).
このとき, 次のように書く:

$$A \leq_m^P B$$

$A \leq_m^P B$ within polynomial time, hardness of $A \leq$ that of B

Theorem 6.1 $A \leq_m^P B$ leads to,

(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in$ co-$\mathcal{NP} \rightarrow A \in$ co-$\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

Note: class $\mathcal{E}$ is exceptional. Generally, $B \in \mathcal{E} \rightarrow A \in \mathcal{E}$ is not true.

Ex.6.2: If we define ONE$\equiv \{1\}$, for each set L in $\mathcal{P}$ we have

$$L \leq_m^P \text{ONE}$$

If we define $h(x) \equiv \begin{cases} 1, & \text{if } x \in L, \\ 0, & \text{otherwise} \end{cases}$

(1) h is a total function from Σ^* onto Σ^*.
(2) $x \in \Sigma^* [x \in L \leftrightarrow h(x) \in \text{ONE}]$
(3) h is polynomial-time computable(so is computation $L \in \mathcal{P} \rightarrow x \in L$)

$A \leq_m^P B$　多項式時間の範囲内では，Aの難しさ $\leq$ Bの難しさ

定理6.1. $A \leq_m^P B$ のとき，

(1) $B \in \mathcal{P} \rightarrow A \in \mathcal{P}$.
(2) $B \in \mathcal{NP} \rightarrow A \in \mathcal{NP}$.
(3) $B \in$ co-$\mathcal{NP} \rightarrow A \in$ co-$\mathcal{NP}$.
(4) $B \in \mathcal{EXP} \rightarrow A \in \mathcal{EXP}$.

補注：クラス$\mathcal{E}$は例外．一般には，$B \in \mathcal{E} \rightarrow A \in \mathcal{E}$とはならない．

例6.2: ONE$\equiv \{1\}$と定義するとき，クラス$\mathcal{P}$のすべての集合Lについて　$L \leq_m^P$ ONE

が成り立つ．　$h(x) \equiv \begin{cases} 1, & x \in L\text{のとき}, \\ 0, & \text{その他のとき} \end{cases}$

と定義すると，(1) hはΣ^*からΣ^*への全域的関数．
(2) $x \in \Sigma^*[x \in L \leftrightarrow h(x) \in \text{ONE}]$
(3) hは多項式時間計算可能($L \in \mathcal{P} \rightarrow x \in L$の判定も多項式時間内)

Theorem 6.2: A, B, C: arbitrary sets

(1) $A \leq_m^P A$

(2) $A \leq_m^P B \wedge B \leq_m^P C \rightarrow A \leq_m^P C$

Def: $A \equiv_m^P B \leftrightarrow A \leq_m^P B \wedge B \leq_m^P A$

$\equiv_m^P$ is an equivalence relation.

定理6.2: A, B, C: 任意の集合

(1) $A \leq_m^P A$

(2) $A \leq_m^P B \wedge B \leq_m^P C \rightarrow A \leq_m^P C$

定義: $A \equiv_m^P B \leftrightarrow A \leq_m^P B \wedge B \leq_m^P A$

$\equiv_m^P$ は同値関係

Relation among satisfiability problems of propositional expressions

2SAT （propositional satisfiability problem）
3SAT
SAT
ExSAT （extended propositional satisfiability problem）

$2\mathrm{SAT} \leq_m^P 3\mathrm{SAT}$

- at most k…trivial
- exactly k…
  - ➢easy if you can repeat the same literal.
  - ➢the other case … good exercise!

Similarly,

$3\mathrm{SAT} \leq_m^P \mathrm{SAT} \leq_m^P \mathrm{ExSAT}$

$2\mathrm{SAT} \leq_m^P 3\mathrm{SAT} \leq_m^P \mathrm{SAT} \leq_m^P \mathrm{ExSAT}$ (6.1)

Here, if we can show

$\mathrm{ExSAT} \leq_m^P 3\mathrm{SAT}$

then we have

$3\mathrm{SAT} \equiv_m^P \mathrm{SAT} \equiv_m^P \mathrm{ExSAT}$

命題論理式の充足可能性問題の間の関係

2SAT　（命題論理式充足性問題：二和積形式）
3SAT　（命題論理式充足性問題：三和積形式）
SAT　（命題論理式充足性問題）
ExSAT　（拡張命題論理式充足性問題）

$$2\mathrm{SAT} \leq_m^P 3\mathrm{SAT}$$

- 高々k個…自明
- ちょうどk個…
  - ➢同じリテラルを使ってよいなら簡単。
  - ➢だめなら…考えてみよう！

同様に,

$$3\mathrm{SAT} \leq_m^P \mathrm{SAT} \leq_m^P \mathrm{ExSAT}$$

$$2\mathrm{SAT} \leq_m^P 3\mathrm{SAT} \leq_m^P \mathrm{SAT} \leq_m^P \mathrm{ExSAT} \quad (6.1)$$

ここで

$$\mathrm{ExSAT} \leq_m^P 3\mathrm{SAT}$$

であることを示せると,

$$3\mathrm{SAT} \equiv_m^P \mathrm{SAT} \equiv_m^P \mathrm{ExSAT}$$

となる.

Ex. 6.3: Reduction from ExSAT to 3SAT

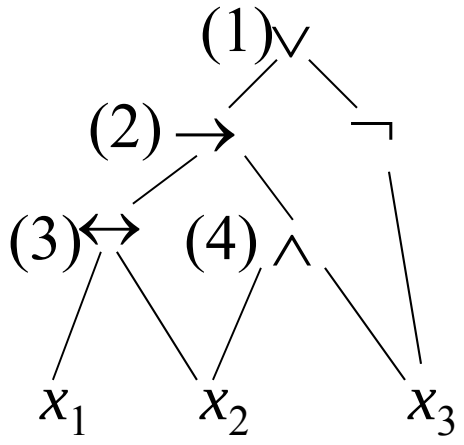
$$E_1(x_1, x_2, x_3) \equiv [[x_1 \leftrightarrow x_2] \rightarrow [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1, x_2, x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \rightarrow U_4]] \wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

Then, [E_1 is satisfiable] $\leftrightarrow$ [F_1 is satisfiable] (6.2)

F_1 is easier to be converted to 3SAT form.

How to construct F_1



$$(1) V_1 \equiv V_2 \vee \neg x_3$$

$$(2) V_2 \equiv [V_3 \rightarrow V_4]$$

$$(3) V_3 \equiv [x_1 \leftrightarrow x_2]$$

$$(4) V_4 \equiv x_2 \wedge x_3$$

To construct $\boldsymbol{F_1}$ we let $V_i \rightarrow U_i$, and connect expressions of V_i by $\wedge$

例6.3: ExSATから3SATへの還元

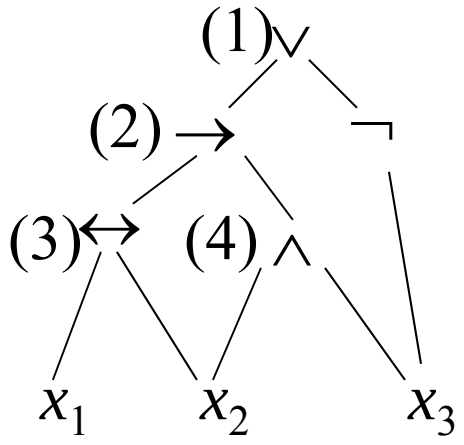
$$E_1(x_1, x_2, x_3) \equiv [[x_1 \leftrightarrow x_2] \rightarrow [x_2 \wedge x_3]] \vee \neg x_3$$

$$F_1(x_1, x_2, x_3) \equiv U_1 \wedge [U_1 \leftrightarrow [U_2 \vee \neg x_3]] \wedge [U_2 \leftrightarrow [U_3 \rightarrow U_4]] \wedge [U_3 \leftrightarrow [x_1 \leftrightarrow x_2]] \wedge [U_4 \leftrightarrow [x_2 \wedge x_3]]$$

このとき，[E_1が充足可能] $\leftrightarrow$ [F_1が充足可能]　　(6.2)

F_1は三和積形式に直しやすい形になっている．

F_1の構成方法



$$(1) V_1 \equiv V_2 \vee \neg x_3$$

$$(2) V_2 \equiv [V_3 \rightarrow V_4]$$

$$(3) V_3 \equiv [x_1 \leftrightarrow x_2]$$

$$(4) V_4 \equiv x_2 \wedge x_3$$

F_1を構成するために，$V_i \rightarrow U_i$とし，V_iの定義式を $\wedge$で結ぶ

From the construction of F_1
(1) F_1 is never true unless each U_i is $V_i(x_1, x_2, x_3)$.
(2) If each U_i is $V_i(x_1, x_2, x_3)$, we have $F_1 = E_1$

The above properties are proved by using induction.

proof is omitted.

Conversion to 3SAT form

$a \rightarrow b = \neg a \vee b$

$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a) = [\neg a \vee b] \wedge [\neg b \vee a]$: useful relations

$$U_1 \leftrightarrow [U_2 \vee \neg x_3] = [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg [U_2 \vee \neg x_2]]$$
$$= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]]$$
$$= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2]$$
$$= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2]$$

Others are similar.
Thus, every 3SAT form is converted.

F_1 の構成方法より，
(1)各U_i の値を$V_i(x_1, x_2, x_3)$としない限り，F_1は真にはならない．
(2)各U_iの値を$V_i(x_1, x_2, x_3)$としたとき，$F_1 = E_1$

上の性質が成り立つことは，帰納法を用いるなどして証明可能．
証明は省略．

三和積形式への変換

$a \to b = \neg a \vee b$

$a \leftrightarrow b = (a \to b) \wedge (b \to a) = [\neg a \vee b] \wedge [\neg b \vee a]$であることを用いる．

$$
\begin{aligned}
U_1 \leftrightarrow [U_2 \vee \neg x_3] &= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg [U_2 \vee \neg x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee [\neg U_2 \wedge x_2]] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2] \wedge [U_1 \vee x_2] \\
&= [\neg U_1 \vee U_2 \vee \neg x_3] \wedge [U_1 \vee \neg U_2 \vee \neg U_2] \wedge [U_1 \vee x_2 \vee x_2]
\end{aligned}
$$

他も同様．
よって，すべて三和積形式に変形できることがわかる．

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Def.6.2: For a class $\mathcal{C}$, if a set A satisfies the following conditions, then it is called $\mathcal{C}$-complete (under $\leq_m^P$)

(a) $\forall L \in \mathcal{C}\ [L \leq_m^P A]$

(b) $A \in \mathcal{C}$

Note : Sets satisfying the condition (a) are called $\mathcal{C}$-hard.

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

定義6.2: 計算量クラス$\mathcal{C}$に対し，集合Aが次の条件を満たすとき，
それを($\leq_m^P$の下で)$\mathcal{C}$-完全という.

(a) $\forall L \in \mathcal{C}\,[L \leq_m^P A]$

(b) $A \in \mathcal{C}$

補注：条件(a)を満たす集合は$\mathcal{C}$-困難.

6.2.Completeness based on Polynomial-time Reducibility

6.2.1. Definition of Completeness and its Basic Properties

Ex.6.5. Examples of $\mathcal{NP}$-complete sets

3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VC, etc

$\mathcal{EXP}$-complete sets

EVAL-IN-E, HALT-IN-E, etc.

EVAL - IN - E :

Input : $< a, x, \bar{t} >$

a : the code of a program with 1 input, $x \in \Sigma^*, \bar{t} \geq 0$

Output : $eval\text{-}in\text{-}time(a, x, \overline{2}^{\bar{t}}) = accept$?

6.2. 多項式時間還元可能性に基づく完全性

6.2.1. 完全性の定義とその基本的性質

例6.5. クラス$\mathcal{NP}$の完全集合の例

3SAT, SAT, ExSAT, DHAM, KNAP, BIN, VCなど

クラス$\mathcal{EXP}$の完全集合

EVAL-IN-E, HALT-IN-Eなど

EVAL-IN-E:

入力: $< a, x, \bar{t} >$

a: 1入力プログラムのコード, $x \in \Sigma^*, \bar{t} \geq 0$

出力: $eval\text{-}in\text{-}time(a, x, \overline{2^{\bar{t}}}) = accept?$

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subset$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

Proof: CP: contraposition

(1) Let B be any $\mathcal{C}$-set. Then, since A is $\mathcal{C}$-hard,

$B \leq_m^P A$ and by the assumption $A \in \mathcal{P}$ we have $B \in \mathcal{P}$ (Th. 6.1)

(2), (3), (4) are similar.

定理6.3. 任意の$\mathcal{C}$-困難集合（含：$\mathcal{C}$-完全集合）Aに対し，

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$　　対偶は $\mathcal{C} \not\subseteq \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$　　対偶は $\mathcal{C} \not\subseteq \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$　　対偶は $\mathcal{C} \not\subseteq$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$　　対偶は $\mathcal{C} \not\subseteq \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

証明：

(1) Bを任意の$\mathcal{C}$集合とすると，Aは$\mathcal{C}$-困難だから，

$B \leq_m^P A$　一方，$A \in \mathcal{P}$の仮定より，$B \in \mathcal{P}$（定理6.1）

(2), (3), (4)も同様

Theorem 6.3. For any $\mathcal{C}$-hard (or $\mathcal{C}$-complete) set A,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ CP: $\mathcal{C} \not\subset \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$ CP: $\mathcal{C} \not\subset \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$ CP: $\mathcal{C} \not\subset$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$ CP: $\mathcal{C} \not\subset \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

Theorem 5.9.

(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP}$

$\rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

Ex.6.6: Meaning of Theorem 6.3 (class $\mathcal{NP}$)

Let A be $\mathcal{NP}$-complete set.

By the contraposition of Theorem 6.3(1) we have

$\mathcal{NP} \neq \mathcal{P} \rightarrow A \notin \mathcal{P}$

By the contraposition of Theorem 6.3(3) and that of Theorem 5.9(1),

$A \notin$ co-$\mathcal{NP}$

That is, $\mathcal{NP}$-complete sets are $\mathcal{NP}$-sets that cannot be recognized in polynomial time unless $\mathcal{P} = \mathcal{NP}$.

定理6.3. 任意の$\mathcal{C}$-困難集合(含:$\mathcal{C}$-完全集合)Aに対し,

(1) $A \in \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$　　対偶は $\mathcal{C} \not\subseteq \mathcal{P} \rightarrow A \notin \mathcal{P}$

(2) $A \in \mathcal{NP} \rightarrow \mathcal{C} \subseteq \mathcal{NP}$　　対偶は $\mathcal{C} \not\subseteq \mathcal{NP} \rightarrow A \notin \mathcal{NP}$

(3) $A \in$ co-$\mathcal{NP} \rightarrow \mathcal{C} \subseteq$ co-$\mathcal{NP}$　　対偶は $\mathcal{C} \not\subseteq$ co-$\mathcal{NP} \rightarrow A \notin$ co-$\mathcal{NP}$

(4) $A \in \mathcal{EXP} \rightarrow \mathcal{C} \subseteq \mathcal{EXP}$　　対偶は $\mathcal{C} \not\subseteq \mathcal{EXP} \rightarrow A \notin \mathcal{EXP}$

例6.6. 定理6.3の意味(クラス$\mathcal{NP}$)

Aを$\mathcal{NP}$-完全集合とする.

定理6.3(1)の対偶より,

$\mathcal{NP} \neq \mathcal{P} \rightarrow A \notin \mathcal{P}$

定理6.3(3)の対偶と定理5.9(1)の対偶より,

$A \notin$ co-$\mathcal{NP}$

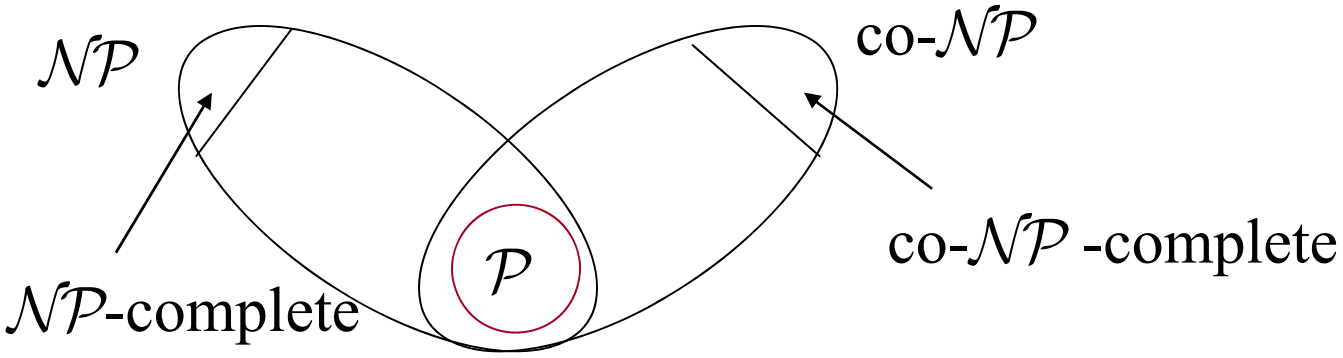
つまり, $\mathcal{NP}$-完全集合は$\mathcal{P} \neq \mathcal{NP}$である限り,

多項式時間では認識できない.

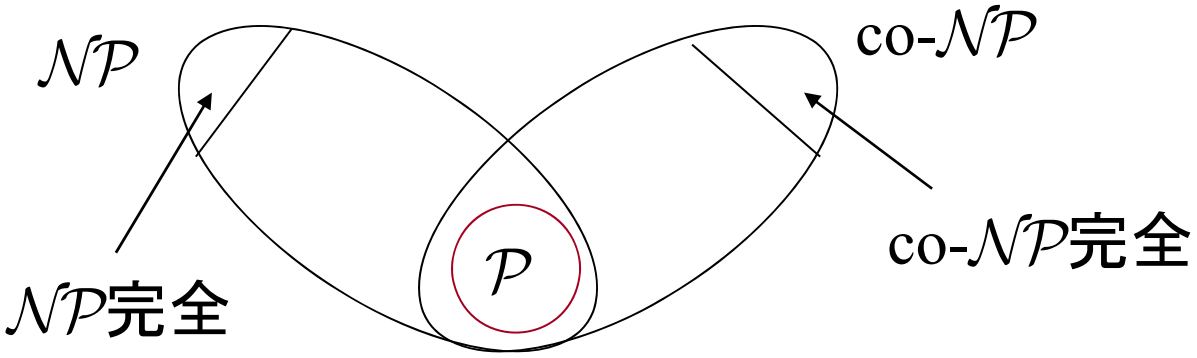
定理5.9.

(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP} \rightarrow \mathcal{NP} =$ co-$\mathcal{NP}$

$\mathcal{NP}$-compete sets are $\mathcal{NP}$-sets that do not belong to $\mathcal{NP} \cap$ co-$\mathcal{NP}$ unless $\mathcal{P} = \mathcal{NP}$.



$\mathcal{NP}$-完全集合は$\mathcal{P} \neq \mathcal{NP}$である限り，$\mathcal{NP} \cap$ co-$\mathcal{NP}$には入らない$\mathcal{NP}$集合である.



Ex. 6.7. Meaning of Theorem 6.3 (class $\mathcal{EXP}$)

Let D be any $\mathcal{EXP}$-complete set.

Contraposition of Theorem 6.3(1)
$(C \notin \mathcal{P} \rightarrow A \notin \mathcal{P}$, where $\mathcal{EXP} \neq \mathcal{P} \rightarrow D \notin \mathcal{P})$
$\mathcal{P} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \neq \mathcal{P}$ $(\because \mathcal{P} \subseteq \mathcal{EXP}) \rightarrow D \notin \mathcal{P}$
Contraposition of Theorem 6.3(2) $(C \neq \mathcal{NP} \rightarrow A \notin \mathcal{NP}$,
Here, $\mathcal{EXP} \neq \mathcal{NP} \rightarrow D \notin \mathcal{NP})$
$\mathcal{NP} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \neq \mathcal{NP}$ $(\because \mathcal{NP} \subseteq \mathcal{EXP}) \rightarrow D \notin \mathcal{NP}$
Contraposition of Theorem 6.3(3) $(C \notin$ co-$\mathcal{NP} \rightarrow A \neq$ co-$\mathcal{NP}$,
here, $\mathcal{EXP} \neq$ co-$\mathcal{NP} \rightarrow D \notin$ co-$\mathcal{NP})$
co-$\mathcal{NP} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \neq$ co-$\mathcal{NP} \rightarrow D \notin$ co-$\mathcal{NP}$
But, by Theorem 5.7, since we know $\mathcal{P} \subsetneq \mathcal{EXP}$, we have $D \notin \mathcal{P}$.

$\mathcal{EXP}$-complete sets are not computable in polynomial time.

例6.7. 定理6.3の意味(クラス$\mathcal{EXP}$)

Dを$\mathcal{EXP}$-完全集合とする.

定理6.3(1)の対偶($C \notin \mathcal{P} \rightarrow A \notin \mathcal{P}$, ここでは$\mathcal{EXP} \neq \mathcal{P} \rightarrow D \notin \mathcal{P}$)

$\mathcal{P} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \neq \mathcal{P}$ ($\because \mathcal{P} \subseteq \mathcal{EXP}$) $\rightarrow D \notin \mathcal{P}$

定理6.3(2)の対偶($C \notin \mathcal{NP} \rightarrow A \notin \mathcal{NP}$,
ここでは$\mathcal{EXP} \neq \mathcal{NP} \rightarrow D \notin \mathcal{NP}$)

$\mathcal{NP} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \neq \mathcal{NP}$ ($\because \mathcal{NP} \subseteq \mathcal{EXP}$) $\rightarrow D \notin \mathcal{NP}$

定理6.3(3)の対偶($C \notin$co-$\mathcal{NP} \rightarrow A \notin$co-$\mathcal{NP}$,
ここでは$\mathcal{EXP} \neq$co-$\mathcal{NP} \rightarrow D \notin$co-$\mathcal{NP}$)

co-$\mathcal{NP} \neq \mathcal{EXP} \rightarrow \mathcal{EXP} \neq$co-$\mathcal{NP} \rightarrow D \notin$ co-$\mathcal{NP}$

ところが定理5.7から $\mathcal{P} \subsetneq \mathcal{EXP}$ であるから, $D \notin \mathcal{P}$.

$\mathcal{EXP}$-完全集合は多項式時間では計算不可能.

Theorem 6.4. A: any $\mathcal{C}$-complete set

For any set B we have

(1) $A \leq_m^P B \rightarrow B$ is $\mathcal{C}$-hard.

(2) $A \leq_m^P B \ \wedge B \in \mathcal{C} \rightarrow B$ is $\mathcal{C}$-complete.

Proof:

By Def. 6.2 $\forall L \in \mathcal{C}[L \leq_m^P A]$

By Theorem 6.2, $L \leq_m^P A \wedge A \leq_m^P B \rightarrow L \leq_m^P B$

Therefore, $\forall L \in \mathcal{C}[L \leq_m^P B]$

That is, B is $\mathcal{C}$-hard.

定理6.4. A: 任意の$\mathcal{C}$-完全集合

すべての集合Bに対し,

(1) $A \leq_m^P B$ →Bは$\mathcal{C}$-困難.

(2) $A \leq_m^P B \ \wedge B \in \mathcal{C}$ → Bは$\mathcal{C}$-完全.

証明:

定義6.2より, $\forall L \in \mathcal{C}[L \leq_m^P A]$

定理6.2より, $L \leq_m^P A \wedge A \leq_m^P B \rightarrow L \leq_m^P B$

したがって, $\forall L \in \mathcal{C}[L \leq_m^P B]$

すなわち, Bは$\mathcal{C}$-困難.

$\mathcal{EXPC} \equiv \{L: L \text{ is } \mathcal{EXP}\text{-complete}\}$

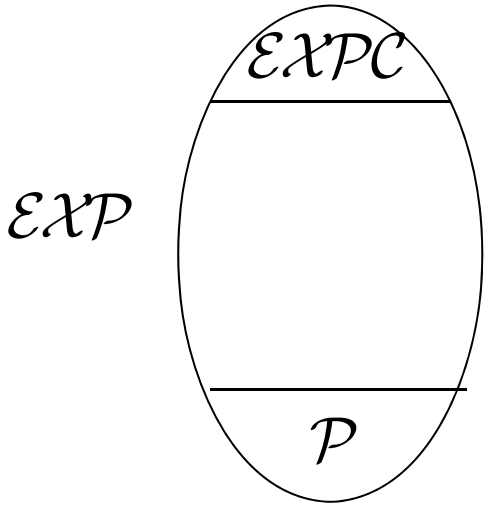
$\mathcal{NPC} \equiv \{L: L \text{ is } \mathcal{NP}\text{-complete}\}$

Then, we have the following theorems.

Theorem 6.5.

(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$

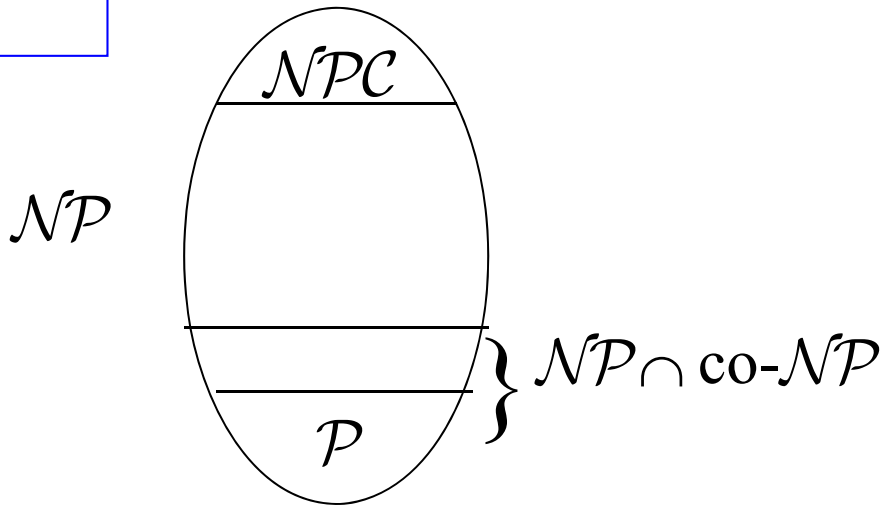
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$



Theorem 6.6: Assuming $\mathcal{P} \neq \mathcal{NP}$

(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$

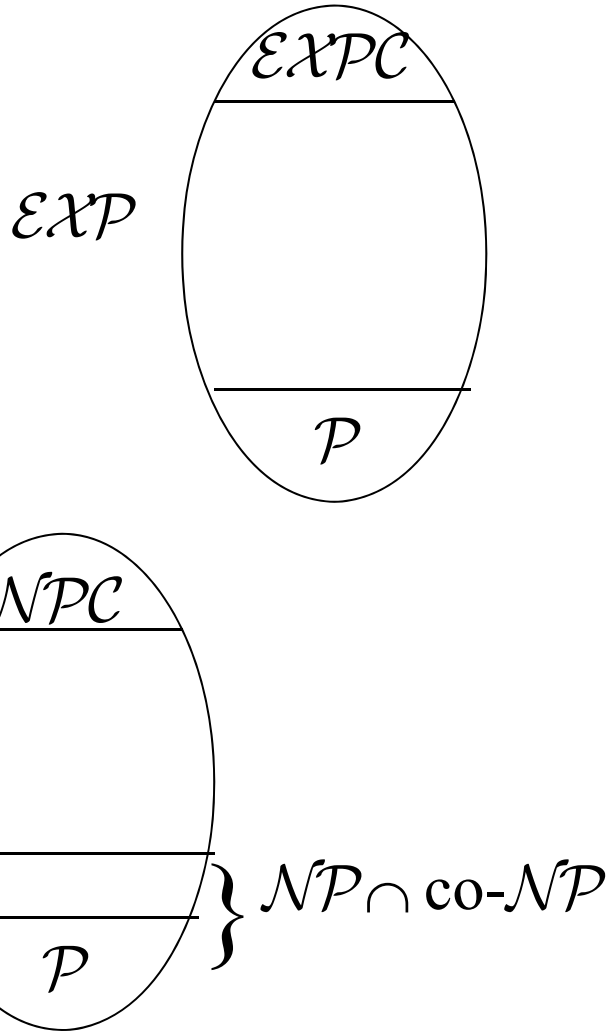
(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$



$\mathcal{EXPC}$ ≡{L: Lは$\mathcal{EXP}$-完全}
$\mathcal{NPC}$ ≡ {L: Lは$\mathcal{NP}$-完全}
とすると，次の定理が成り立つ．

定理6.5.
(1) $\mathcal{EXPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{EXP} - (\mathcal{EXPC} \cup \mathcal{P}) \neq \phi$

定理6.6: $\mathcal{P} \neq \mathcal{NP}$を仮定すると
(1) $\mathcal{NPC} \cap \mathcal{P} = \phi$
(2) $\mathcal{NP} - (\mathcal{NPC} \cup \mathcal{P}) \neq \phi$



Schedule(残りの予定)

- 10/25(Mon):
  - Submission of the report (レポート提出)
- 10/28(Thu): Last class (前半最後の講義)
  - Course Evaluation Questionnaire（授業アンケート）
  - Misc.（その他）
- 11/ 1(Mon): mid-term exam (中間試験)
  - 40 points
  - ✖Textbook, Copy, Printout,…
  - You can bring your own hand-written notebook (手書きノートのみ持ち込み可)
  - Lesson 3~Lesson 6（講義3~講義6）