Abstract:Notes on vertex atlas of planar Danzer tiling

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In 1982 a quasi-crystal with 5-fold rotational symmetry was discovered by Shechtman et al. The most famous 2-dimensional mathematical model for the quasi-crystal may be the Penrose tiling with 5-fold rotational symmetry. In addition, there are the Ammann-Beenker tiling with 8-fold rotational symmetry and the Danzer tiling with 7-fold rotational symmetry(cf.[3]) in typical tilings. Such tilings are called nonperiodic tilings.

We prepare several basic definitions. A planar tiling $\mathcal{T}$ is a countable family of polygons T_i called tiles: $\mathcal{T} = \{T_i \mid i = 1, 2, \cdots\}$ such that $\bigcup_{i=1}^{\infty} T_i = \mathbf{R}^2$ and Int $T_i \cap$ Int $T_j = \emptyset$ if $i \neq j$, where $\mathbf{R}^2$ denotes the 2-dimensional Euclidean space. A configuration (without gap and overlapping) of tiles around a vertex in a tiling is called vertex atlas.

Let $\mathcal{S} = \{S_1, S_2, \cdots, S_l\}$ be a finite set of polygons. When each tile T in a tiling $\mathcal{T}$ is congruent to some $S_i \in \mathcal{S}$, $\mathcal{S}$ is called a prototile set of $\mathcal{T}$. A set of matching rules for a prototile set $\mathcal{S}$ is a finite set of patches that may appear in the tilings admitted by $\mathcal{S}$. Fix $\lambda(> 1)$. For a prototile set $\mathcal{S}$, any prototile is decomposed into λ^{-1} scale-down copies of $\mathcal{S}$. This decomposition is called a substitution rule of $\mathcal{S}$ if such a decomposition is possible. We can construct nonperiodic tilings with a given prototile set by the up-down generation using substitution rule (cf.[1]).

The prototiles of Danzer tiling are six types of triangles with arrows on the edges (three triangles a, b, c in Figure 1 and their reflections).

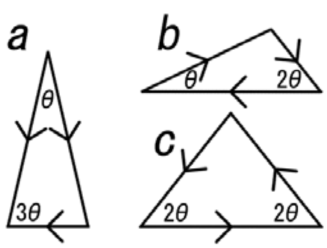

Figure 1: Three prototiles with arrows of Danzer tiling ($\theta = \pi/7$)

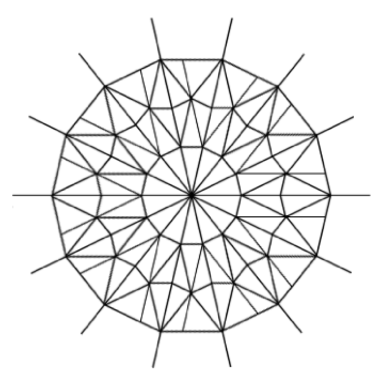

Figure 2: The Danzer tiling with 7-fold rotational symmetry (erasing arrows)

We can construct the Danzer tiling with 7-fold rotational symmetry in Figure 2 using the up-down generation and reflection and rotation(cf.[2],[3]). This note is motivated by the following remark in the appendix of [3]: "29 kinds of vertex atlases appear in Danzer tiling, and these vertex atlases may serve as a matching rule." We study details of his remark, and meet a lot of strange things. We state these results comparing Danzer tiling (DT, in short) with Penrose tiling (PT, in short).

(1) What kinds of vertex atlases ?

PT: (well-known) 8 kinds of vertex atlases with arrows appear in Penrose tilings constructed only by the up-down generation procedure.

DT: In Danzer tilings constructed only by the up-down generation procedure, 39 kinds of vertex atlases with arrows appear, and 29 kinds of vertex atlases appear by erasing arrows.

(2) rotational symmetry and up-down generation:

PT and DT with rotational symmetry cannot be constructed only by the up-down generation procedure. It is necessary to expand to whole plane by using reflection and rotation.

The set of tilings has the canonical metric, called tiling metric.

(3) A limit of sequence of tilings :

PT: The Penrose tiling with 5-fold symmetry is obtained as a limit of sequence of tilings constructed only by the up-down generation procedure.

DT: The Danzer tiling with 7-fold symmetry cannot be obtained as a limit of sequence of tilings constructed only by the up-down generation procedure.

(4) Matching rule:

PT: 8 kinds of vertex atlases with arrows serve as a matching rule in the set of tilings constructed by the up-down generation and reflection and rotation.

DT: 39 kinds of vertex atlases with arrows serve as a matching rule in the set of tilings constructed by the up-down generation, but cannot do in the set of tilings constructed by the up-down generation and reflection and rotation.

It seems that the following questions are open:

(a) For which n tilings with n-fold symmetry can be constructed only by the up-down generation procedure ? (Of course, for $n = 4, 6$ we have the trivial and boring example.)

(b) What is the explicit procedure for constructing tilings as limit ? (We need reflection and rotaition when we construct tilings with rotational symmetry in Penrose tiling.)

References

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